Discrete Games in Endogenous Networks: A Model of Network Effects on Consumer Choices.*

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Abstract. This paper studies socioeconomic networks where agents make both decisions regarding network links and decisions regarding individual actions. In particular, I propose a behavioral model of network formation, where the network configuration arises as the cumulative outcome of agents' optimal decisions over time rather than as the instantaneous outcome of a one-shot game with many players. Alternative protocols of network formation, sequential and k-coalition stage game, are discussed and it is shown that over time the network is absorbed in an equilibrium. Thus this paper contributes the discussion of how Nash (or in general any) equilibria come into being.

Multiplicity of equilibria is a common problem in discrete games of complete information, which have been used to visualize a variety of contexts such as market entry, technology adoption, and peer effects. I develop an estimable specification of the framework, which allows me to deal with this problem in a novel way. In particular, the behavioral assumptions of the model naturally assign a probability to each equilibrium that can be used to form an integrated likelihood.

I first discuss properties of a generic estimable specification of the model and then derive a closed-form expression for the likelihood. This presents an identification argument for model's parameters and facilitates the estimation (say via Bayesian MCMC algorithm or simulated ML estimator). I use the likelihood to argue that this framework is robust to different techniques for capturing the strategic aspect of individuals' behavior i.e. the sequential model of network dynamics is observationally equivalent to, what is termed as, (random k) k-player stage game dynamics. This is important in that it demonstrates that the model predictions do not hinge on ad hoc assumptions. In addition, the sequential framework is computationally easier to simulate and estimate with an appropriate data set.[1] Finally, the link between the theoretical and the estimable model is discussed from the perspective of the notions of quantile response equilibrium and correlated equilibrium (McKelvey and Palfrey (1995), and Aumann (1974)).

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[1] In a related work Badev (2012) applies this methodology to study how the network position affects consumer choices, and in general how the network mediates the marketing mix, in the context of teen cigarette smoking.

1 Introduction

A substantial body of research in economics, demography, sociology, and social anthropology examines how social interactions shape individual behavior and, more specifically, how peer networks mediate individual choices.[2] For example, peers are potentially very important in studying teen risky behavior, such as decisions to smoke and/or use drugs. Progress in estimating rich social interaction models has been difficult due to the lack of appropriate data sets and the complexity of models involving peer interactions and decisions.

Typically models of social interaction visualize a situation where agents' payoffs depend on the actions of other agents. These situations have been formalized as discrete games - extension of discrete choice models allowing for strategic interdependence of agents payoffs. Discrete games have been used in a variety of contexts including firms market entry decisions (Bresnahan and Reiss (1990); Berry (1992)), firms product quality choice (Mazzeo (2002)), and individual decisions in the presence of peer effects (Brock and Durlauf (2001a); Bisin et al. (2011)) amongst others.

This paper develops a novel approach to modeling discrete games of complete information in the presence of endogenous networks. Although the model is casted in the settings of a social network, the methodology presented here is applicable to a general context where the network structure plays a role in affecting agents' decisions and determining the outcome. Examples of such contexts from the network literature include modeling information diffusion, trade and exchange of goods in non-centralized markets, and providing mutual insurance.[3]

I propose a framework for studying socioeconomic networks, where agents make both decisions regarding (forming and severing) network links and decisions regarding individual actions. This setup is in contrast to extant research where the two decisions are considered separately.[4] My framework introduces three alternative notions of equilibrium network states: Nash, sequentially-stable, and coalition-stable states. I establish existence and discuss properties of the different equilibrium concepts through an extension of the construct of potential function to unweighted bi-directional networks (Rosenthal (1973); Monderer and Shapley (1996)).

A Nash state is defined as the outcome of a one-shot simultaneous game where the players choose from their unrestricted strategy space. Stable states are those robust to certain class of deviations.[5] For $1 < k < n$, k-coalition stable state is defined as a network configuration consistent with the outcome of a Nash equilibrium play of all k-player games, where the strategy sets consists of forming and severing links to the participants of the game and taking an action. The concept of k-coalition stability imposes far less restrictions in terms of coordination between players then the Nash equilibrium. In a k-coalition stable network any k players are playing Nash equilibrium but not necessarily the whole population.

[2]The reader is referred to Manski (1993); Brock and Durlauf (2001a,b); Durlauf (2001), the more recent work of De Giorgi et al. (2010); Lavy and Schlosser (2011); Richards-Shubik (2011) and the surveys in Jackson (2005, 2008); Blume et al. (2010).

[3]For more on the topic see Jackson (2005, 2008).

[4]See Nakajima (2007); Currarini et al. (2009); Calvo-Armengol and Ilkilic (2009); Mele (2010).

[5]Jackson and Wolinsky (1996) first propose the notion of stability in the context of a model where agents can form and severe links only. They discuss the case of $k = 2$ i.e. pairwise stability. The reader is referred to Jackson (2005) for discussion of extensions in models without action choice.

Next, I develop a model of network formation, where the network configuration arises as the outcome of agents optimal behavior over time rather than as the instantaneous outcome of a one-shot game with many players. This approach addresses concerns in the literature on social interactions (and discrete games of complete information more generally) such as how (or why) a given equilibrium is selected among the set of all potential equilibria. Indeed the remark of Kandori, Mailath and Rob (1993) on the Nash equilibrium concept is applicable for any equilibrium notion:

> "While the Nash equilibrium concept has been used extensively in many diverse contexts, game theory has been unsuccessful in explaining *how* players know that a Nash equilibrium will be played. Moreover, the traditional theory is silent on how players know *which* Nash equilibrium is to be played if a game has multiple equally plausible Nash equilibria."

In my model the equilibrium states are absorbing - once reached the system stays there forever. Moreover, for large t the network state is almost surely an element of the set of stable equilibria. I also discuss alternative protocols of network formation, sequential and k-coalition stage game, and demonstrate properties of their dynamics.

To guide the empirical implementation of the model, I develop an estimable specification of the framework capable of handling multiplicity of equilibria. This is a common problem in discrete games of complete information and extant approaches to deal with it include: modeling the outcomes that are unique even in the presence of multiplicity[6] (Bresnahan and Reiss (1990); Berry (1992)), exploring the temporal structure of the play (Berry (1992)), assuming an ad-hoc equilibrium selection rule (Hartmann (2010)), obtaining the identified set estimates (Tamer (2003); Ciliberto and Tamer (2009)), empirically estimating an equilibrium selection rule (Bajari et al. (2010)), and conditioning on the equilibrium selection rule (i.e. being agnostic about the equilibrium selection mechanism; Bisin et al. (2011)).[7] This work is in contrast to the extant approaches in that it models explicitly how the equilibrium comes into being and more specifically the equilibrium selection process; The behavioral assumptions of the model naturally assign probabilities to all equilibria that can be used to form an integrated likelihood.

The properties of a generic estimable specification are discussed and a closed-form expression for the likelihood is obtained. This is advantageous in that it readily presents an identification argument for model parameters and facilitates the estimation (say via Bayesian MCMC algorithm or simulated ML estimator). I use the likelihood to argue that this framework is robust to different techniques for capturing the strategic aspect of individuals behavior i.e. the sequential model of network dynamics is observationally equivalent to, what is termed as, (random k) k-player stage game dynamics. That is to say that the model where every period one individual plays a *best response* induces the same stationary distribution over the space of network configurations $\mathbf{S}_n$ as the model where every period a *coalition of random size and composition* acts

[6]The problem of multiplicity of equilibria has been pointed much earlier than the first proposed solutions. See for example Heckman (1978)).

[7]In a recent work Narayanan (2011) takes a Bayesian perspective to the problem of multiplicity. There the prior reflects the analysts uncertainty about equilibrium selection. His approaches goes beyond estimating the identified set in that the posterior may localize high density regions within the interval estimates and whence provide more information about the parameters.

according to a Nash play. This is important in that it demonstrates that the model predictions do not hinge on ad hoc assumptions. In addition, the sequential framework is computationally easier to simulate and estimate with an appropriate data set.[8]

The rest of the paper is organized as follows. Section 2 lays out the setup and presents the two major sides of the model separately - the problem of peer group selection and the problem of individual choices in the presence of network effects. Section 3 develops a coherent framework to study the network structure in its integrity. There I introduce a set of equilibrium notions - Nash, sequentially-stable and coalition-stable networks. Section 4 introduces the model's dynamics, discusses its properties, and outlines generic estimation strategies. The main conclusion from section 5, is that the framework is robust to different techniques for capturing players incentives to act strategically. There I introduce the (random k) k-coalition model and show that it is observationally equivalent to the sequential model. Finally, section 6 discusses the link between the theoretical and the estimable model from the perspective of the notion of quantile response equilibrium introduced by McKelvey and Palfrey (1995) and the notion of correlated equilibrium introduced by Aumann (1974).

2 Setup and notation

Consider a finite-size pool of peers $I = \{1, 2, ..., n\}$, where every individual[9] is identified by a set of exogenous characteristics X_i. In addition, individuals make a binary choice $a_i \in \{0, 1\}$. The collection of all student cohorts in given high school at a given time period is a good example of such a pool. Another example is the population of geographically isolated area.[10] Peers' attributes X_i may include age, race, gender, belonging to a grade-class, etc and the action a_i may refer to any binary choice such as the decision to smoke, to commit a crime, or to purchase a visible good (iphone, clothing, and jewelry) for example.

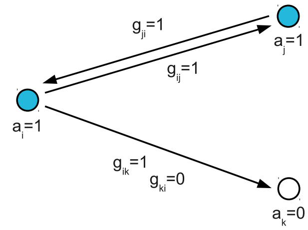
In addition to actions and attributes, individuals feature relationships which form a social network; the individuals are tied by interdependencies, about which we will think as friendships. The *relationship* between peers i and j is summarized in the binary variables g_{ij} and g_{ji} (see figure 1). In particular, if i nominates j as a friend then $g_{ij} = 1$ (i forms a one directional link to j). The relationship between peers does not need to be symmetric (individual i may nominate j as a friend but not vice versa). Such is the case of agents i and k in figure 1. There $g_{ik} = 1$ and $g_{ki} = 0$ so that i regards k as a friend, but the converse is not true. Finally $a_i = 1$ denotes the

[8]In Badev (2012), I apply this methodology to study teen cigarette smoking. There I confirm the evidence of Bisin et al. (2011), who study the problem in the settings of *exogenous* friendship networks, that the peer effect contains smoking i.e. in the presence of randomly assigned friends smoking prevalence is lower. Furthermore, my model goes beyond in quantifying the role of homophily, and more generally the network endogeneity, in the policy mechanisms for controlling teen risky behaviors.

[9]The terms agent, individual, peer and later player are used interchangeably to denote a generic decision-maker capable of creating and severing directed links and committing actions.

[10]Any closed collection of individuals, who draw fiends within themselves will fit the assumptions of the model. Alternatively this assumption (if the data does not span the complete network for example) can be relaxed if one conditions on the existing friendships to outsiders of the pool. Peers who are not in the consideration pool are not part of the model, in the sense that, the links to them can be thought of as part of the fixed attributes X_i. Then the model will explain the formation of new friendships within the data pool conditioning on the ones with outsiders.

Figure 1: Example of network configuration with $n = 3$ individuals - i,j, and k.



state where peer i has committed to some action from a binary discrete set $\{0, 1\}$. The model in this paper does not rely on any specific assumptions, about the type of actions, other than being a binary choice that is potentially related with the friendship network. At any point in time the network configuration is completely defined by the state of all *edges* g_{ij} and *vertices* a_i. Formally the network state is given by:

$$G = \left(\{g_{ij}\}_{i,j=1}^{n}, \{a_i\}_{i=1}^{n} \right).$$

Let $\mathbf{S}_n$ be the set of all possible (directed and unweighted) n-node networks. It is worth noting that this is a large set - $|\mathbf{S}_n| = 2^{n^2}$ and grows exponentially in the size of the network n.[11] Finally it will be convenient to denote the state of the network including all edges and vertices but one edge with G_{-ij}. Similarly G_{-i} fixes the network state excluding vertex (the action of agent) i.

2.1 Two Examples of Behavioral Models of Network Formation

To illustrate the mechanics of the model in the next section, I first start with two special cases that have been discussed in the literature.[12] To set up the stage consider a behavioral model of network formation where in the beginning of every period t (time is discrete) a single agent is drawn from the pool and given the opportunity *either* to commit an action *or* to form a friendship (draw a link to another individual). These two decision modes give rise to different models capable of addressing distinct issues. I present these models in turn.

First, consider the case of exogenously given *fully-connected* network. This setup can be related to a situation of exogenously formed peer groups as is the case of high school classes. In every period one (randomly chosen) individual makes a choice $a_i \in \{0, 1\}$, given the choices of her peers according to utility function:

$$U_i \left(a_{i,t} = 1 | G_{-i,t-1}, X \right) = v \left(X_i \right) + \sum_{j \neq i} a_{j,t-1} h \left(X_i, X_j \right) \tag{1}$$

with $U_i \left(a_{i,t} = 0 | G_{-i,t-1}, X \right)$ normalized to zero for all t and i, and $X = (X_1, ..., X_n)$. The first term above reflects the utility of committing the action, which is allowed to depend on

[11] For example for $n = 10$, $|\mathbf{S}_n| = 2^{100}$.

[12] The first model is due to Nakajima (2007) and the second to Mele (2010).

the individual's attributes X_i. The second term - the summation indexed by j - captures the peer effects; Note how the strength of the peer effects is allowed to vary with the individuals attributes X_i, X_j, so that the influence between individuals is heterogeneous. Over time the sequence of optimal choices in the population generates a sequence of network configurations, whose stationary distribution reflects a situation where the agents are given the ability to revise their choices frequently, taking into account the choices of the others.

Another special case is the one where peer actions are taken to be exogenous. That is given the peer characteristics and actions one can study how agents link (choose friends) in social networks. In this version, at each period one (randomly drawn) individual decides to befriend or not another individual (who she randomly meets) from a fixed population of potential friends. Such a friendship brings the decision maker i utility which may depend on her characteristics X_i, the characteristics of the individual she met j - X_j and the state of their relationship. Formally

$$U_i\left(g_{ij,t}=1|G_{-ij,t-1},X\right)=w_{ij}\left(X_i,X_j\right)+g_{ji,t-1}m\left(X_i,X_j\right) \tag{2}$$

As before $U_i\left(g_{ij,t}=0|G_{-ij,t-1},X\right)=0$ for all i, j, and t. Note the flexibility of this specification - i's utility of befriending j varies with the attributes X_i, X_j and whether or not their friendship is reciprocal (the second term).

3 A Model of Peer Selection and Individual Actions

This section presents a unified framework that can accommodate both the problem of peer selection and the problem of optimal action choice. The framework is based on a behavioral model of utility maximization and random matching. The discrete time infinite horizon setup reflects the assumption that individuals can frequently revise their actions and friendship decisions.

3.1 Model Description

Matching process. In every period a randomly chosen agent (say i) faces one of two types of choices. Either she can decide whether or not to commit a binary action $a_i\in\{0,1\}$ or she can befriend a peer who she randomly meets. The rules of the game are determined by the outcome of the *matching* process - a stochastic process $\{\mu_t\}_{t=1}^{\infty}$ which maps the state of the network to probability distribution over ordered pairs of indexes

$$\mu_t:\mathbf{S}_n\times\mathbf{X}_n\longrightarrow\Delta\left(I\times I\right)$$

where $\mathbf{X}_n$ is the space of all possible attributes of population of size n. An individual i meets j with probability:[13]

$$\Pr\left(\mu_t=(i,j)\,|G_{t-1},X\right)=\mu_{i,j}\left(G_{t-1},X\right) \tag{3}$$

[13] With some slight abuse of notation I use μ to denote a random variable as well as its realization as well as the probability distribution of this same random variable. More precision will not be illuminating while it will proliferate the notation.

Clearly $\sum_{i,j} \mu_{i,j}(G, X) = 1$ for all $G \in \mathbf{S}_n$ and $X \in \mathbf{X}_n$. The simplest technology is where any meeting is equally probable $\mu_{i,j}(G_{t-1}, X) = \frac{1}{n^2}$. However (3) is general enough to accommodate meeting technology with bias for people with similar actions and/or characteristics as for example $\mu_{i,j}(G_{t-1}, X) \propto \exp\{-d(a_i, a_j) - d(X_i, X_j)\}$ with d a generic distance function (see the discussion in Mele (2010)).

Actions. In case $\mu_t = (i, i)$, agent i faces the choice of action a_i. The optimality of her action hinges on her friendships links and an *endogenous peer norm* (the second term below). The utility of one of the alternatives (here $a_i = 0$) is normalized to zero while the other is given by:

$$U_{1,i}(a_{i,t} = 1 | G_{-i,t-1}, X) = v(X_i) + \underbrace{\sum_{j \neq i} a_{j,t-1} g_{ij,t-1} g_{ji,t-1} \phi_{ij}}_{\text{endogenous peer effect}} + \underbrace{\sum_{j \neq i} a_{j,t-1} h(X_i, X_j)}_{\text{aggregate peer effect}} \tag{4}$$

Here the utility of choice $a_{i,t} = 1$ may depend on the individual attributes - $v(X_i)$, as well as a *composite* peer norm (the summation terms). Friends are allowed to have a direct influence on individual's utility - terms $a_{j,t-1} g_{ij,t-1} g_{ji,t-1} \phi_{ij}$ in the first summation. That is one is influenced strongly by the behavior of her friends as opposed to casual individuals. On the other side, the function $h(.,.)$ in the last summation captures the aggregate peer norm. Note how the aggregate peer norm is allowed to depend on peers' attributes X_j.

Friendship formation. If in period t agent i has met a peer j (i.e. $\mu_t = (i, j)$), i can decide to befriend j or not. In the spirit of random utility models her friendship brings her utility

$$U_{2,i}(g_{ij,t} = 1 | G_{-ij,t-1}, X) = w(X_i, X_j) + \underbrace{g_{ji,t-1} a_{i,t-1} a_{j,t-1} \psi_{ij}}_{\text{choice segregation}} + g_{ji,t-1} m(X_i, X_j) \tag{5}$$

as before one needs the normalization $U_{2,i}(g_{ij} = 0 | G_{-ij}, X) = 0$. The first term $w(X_i, X_j)$ captures the incentives of i to befriend j, which may depend on their degree of similarity. The second term in the utility function deserves special attention. It reflects the degree of similarity in the choices of i and j, which is allowed to create addition stimulae for i to nominate j as a friend. Thus action similarity (i.e. if $a_{i,t-1} a_{j,t-1} = 1$) has the potential to influence i's decision to befriend j. Finally, the last term captures the incentives of i to befriend j due to the reciprocity of their friendship.

The time-dimension of the model deserves additional attention. The main assumption is that individuals make their choices sequentially (for alternative - observationally equivalent - protocol see below) and can revise these decisions frequently. The sequential framework coupled with the best response dynamics of Blume (1993, 2003) implies that time periods do not need to be of equal length. Rather, different periods index instances of individual decisions which aim at maximizing the instantaneous utility. In rationalizing the individual's responses, the model puts all the weight on the contemporaneous environment rather than on the human ability to forecast the actions of the other. This is an adequate visualization of the reality under at lease two scenarios. First, if the action of a single individual cannot affect substantially the future development of the network. In such cases individuals do not need to act strategically with respect to the future development of the network. Second, if individuals form stationary expectations about the future which are, in fact, consistent with the current network state. In this case, although individuals need to act strategically with respect to the future, their best response to the current state is indeed a best response to the expected network state tomorrow as well. Thus the proposed approach is likely to be plausible in studying decisions such as smoking and drinking, while it seems inappropriate in situations where individual's forecast about the future is likely to be important (for example to study fertility or career choices).

3.2 A Game with Potential

This section introduces Rosenthal (1973) and Monderer and Shapley (1996) notion of potential function to the analysis (for more on the origin of the idea see the remark below). In the present settings, the potential will bear similarities with what is called the value of a graph (Jackson and Wolinsky (1996)). It turns out that this is perspective will present an opportunity to treat the dynamics of the network and the different notions of stability in an uniform way. The following assumption will allow me to use a single function on the network state - the potential - to characterize the incentives of the players at their turns to make a move.

Assumption 1 *For the functions h and m in (4) and (5)*

$$\begin{aligned} h(X_i, X_j) &= h(X_j, X_i) \\ m(X_i, X_j) &= m(X_j, X_i) \end{aligned}$$

hold for any $i, j \in I$. Furthermore for the coefficients ϕ and ψ

$$\phi_{ij} = \phi_{ji} = \psi_{ij} = \psi_{ji}$$

holds for any $i, j \in I$.

Assumption 1 places two types of restrictions. The first type is the symmetry of peer effects.[14] The magnitude of peer influence from agent i to j equals to the one from agent j to i. Under this heading are the restrictions on $h(.,.)$, $m(.,.)$ and that $\phi_{i,j} = \phi_{j,i}$ and $\psi_{i,j} = \psi_{j,i}$. Importantly, however the symmetry assumption does not imply that the overall peer effect is the same between individuals who are friends. Indeed two friends do not need to have the same network of friends and hence the same peer pressure on their decisions.

The second normalization is that peer influence from actions to friendships has to be the same as from friendship to actions - $\phi_{i,j} = \psi_{j,i}$. Given that two two individuals have decided to become (mutual) friends and have committed to an action choice, one cannot separately identify which decision came first - to become friends or to commit the action.

With assumption 1 the following proposition holds (the proof can be found in the appendix):

Proposition 1 *With assumption 1, the network formation game has a* potential function $\mathcal{P}$:

[14] This restriction is somewhat common in the related literature (see Nakajima (2007); Mele (2010) and is also encountered in the literature on qualitative response models (Heckman (1978); Amemiya (1981)).

$\mathcal{G}_n \times \mathbf{X}_n \longrightarrow \mathbb{R}$:

$$\mathcal{P}(G,X) = \sum_i a_i v(X_i) + \frac{1}{4}\sum_i\sum_j a_i a_j g_{ij} g_{ji}\phi_{ij} + \frac{1}{2}\sum_i\sum_j a_i a_j h(X_i,X_j) + \sum_i\sum_j g_{ij} w\left(X_i,X_j\right) + \frac{1}{4}\sum_i\sum_j a_i a_j g_{ij} g_{ji}\psi_{ij} + \frac{1}{2}\sum_i\sum_j g_{ij} g_{ji} m\left(X_i,X_j\right) \quad (6)$$

In particular the network formation game is a potential game.

The content of proposition 1 is straightforward. At any point in time the incentives of a player to act in a certain way are completely characterized by the potential function i.e. the potential and the utility functions (4) and (5) define the same preference relation over the network configuration.[15]

The conclusion of proposition 1 is very important. That the choice game has a potential precludes cyclical behavior and guarantees that equilibrium (to be defined more precisely) exists. Furthermore the potential function enables the analytical derivation of the likelihood and makes the identification of the model transparent. Finally, proposition 1 helps us understand the generality of the decision protocol. In particular, it is useful in demonstrating that the sequential dynamics and the (random k) k-player dynamics induce the same stationary distribution over $\mathbf{S}_n$ (see theorem 9 and theorem 10).

Remark. Rosenthal (1973) is the first to use potential functions to prove the existence of a pure-strategy Nash equilibrium for the class of congestion games. Monderer and Shapley (1996) discuss several notions of potential functions for games in strategic form and obtain characterization of potential games. (see their paper for more references to early work on potential games) However the construct of potential functions originates in physics where it has been used for some time to reduce the complexity of certain type of problems (see harmonic functions for more).

3.3 Equilibrium Network States

This section introduces three notions of network equilibrium states: Nash, coalition-stable, and sequentially-stable network states. Coalition-stability is a natural generalization of the notion of pairwise-stability, proposed by Jackson and Wolinsky (1996).[16] Both coalition-stability and sequential-stability are weaker notions than that of Nash networks, a point that will be clarified by proposition 3.

Broadly speaking stable states are states that are robust to certain class of deviations. Importantly, however, coalition-stability[17] and sequential-stability are network equilibrium concepts that dispense from any behavioral model of network formation. While there is some

[15] For more details on this see the proof.

[16] Jackson and Wolinsky (1996) introduce pairwise-stability in undirected networks of only links (i.e. without action choice). In their settings the agents are allowed to form and sever links. Following their paper, the notion of pairwise-stability has been extended to include extra considerations similar to the ones here, for example allowing agents to form or sever many links at once. See the discussion in Jackson (2005).

[17] Including pairwise-stability as a special case.

argument for the state to persist once it is reached, it is not clear what behavioral mechanism will drive the system to such a state. Moreover it is not clear if these states are stable under a perturbation not in the neighborhood of an equilibrium. I return to this issue later.

3.3.1 Nash Network States

Define a Nash network state as the outcome of a Nash equilibrium play in the network formation game, where all agents choose from their unrestricted strategy spaces. In particular, with the set of players given by I, the strategy space is $S_i = \{0,1\}^n$ with typical element:

$$G_{(i)} = (a_i, \{g_{ij}\}_{j \neq i}) \in S_i$$

The above is a description of the one-shot game where each player announces which friends to nominate and what action to take. The payoff function is $u_i : \mathcal{G}_n \to \mathbb{R}$

$$\begin{aligned} u_i(G) \quad = \quad & a_i v(X_i) + a_i \sum_j a_j h(X_i, X_j) + \sum_j g_{ij} g_{ji} a_i a_j \psi + \\ & \sum_j g_{ij} w(X_i, X_j) + \sum_j g_{ij} g_{ji} m(X_i, X_j) \end{aligned} \tag{7}$$

The content of the payoff function is intuitive. In fact, it exactly summarizes the incentives embedded in (4) and (5). By that I mean that (7) induces the same preference relation as (4) and (5). Indeed it is easy to verify that one can substitute (7) for both (4) and (5) and our results will not change. The only difference is that (7) is defined on the whole $\mathbf{S}_n$.

Denote the network configurations which arise from a Nash equilibrium play as $S^*_{NE} \subset \mathbf{S}_n$. With that the following theorem holds:

Corollary 2 *With assumption 1, for the simultaneous-move one-shot network formation game:*

1. *There exists a Nash equilibrium.*
2. *The set of all Nash equilibrium outcomes is completely characterized by the following property of the potential function:*

$$S^*_{NE} = \left\{ G \in \mathbf{S}_n : G_{(i)} \in \arg \max_{G'_{(i)} \in S_i} \mathcal{P}\left(G'_{(i)} | G_{(-i)}\right), \forall i \in I \right\} \tag{8}$$

In essence the conclusion above follows the argument of Rosenthal (1973). It relies on the fact that the game has a potential.[18]

3.3.2 Stable Network States

This section extends the notion of stability for states in a bi-directional networks with edges and vertices. Typically a network state is considered as stable if it is robust to certain deviations. In

[18] A detailed proof is available upon request.

particular, define a network state as *k-coalition-stable*, for $1 < k < n$, if no coalition of k peers have incentive to chose another Nash equilibrium in the k-player game where each participant chooses actions and directed links simultaneously. More formally, a k-coalition stable network configuration is consistent with the outcome of the Nash equilibrium play[19] of the following game:

1. The set of players is $\{i_1, i_2, ..., i_k\}$
2. The strategy space is $S_{i_r} = \{0,1\}^k$, for $r \in \{1, ..., k\}$, with typical element

$$s_{i_r} = \left(a_{i_r}, \{g_{i_r i_{r'}}\}_{r,r'=1}^{k}\right) \tag{9}$$

   where $r \neq r'$.
3. As above, the payoffs are given by (7).

The above definition relaxes the excessive rationality assumptions implied by the Nash equilibrium, however it preserves some coordination between players. Each k-tuple of players is required to play Nash equilibrium of the smaller k-player game. Moreover in case multiple equilibria are present, the one with the largest potential is played. These assumptions are somewhat milder, yet one may argue that the larger is k the more coordination is expected from the players. Now I define an even weaker notion of stability.

Let $G \in \mathbf{S}_n$ be *sequentially-stable* if no individual has incentive to alter exactly one of her decisions - i.e. either change a_i or g_{ij} for some j. A moment of reflection shows that a network state is sequentially stable if and only if it is the outcome of the following game:

1. The set of players is I.
2. The strategy space is $S_i = \{0,1\} \cup (I_{-i} \times \{0,1\})$ with typical element s_i which is either a_i or (j, g_{ij}) for some $j \neq i$.
3. As above, the payoffs are given by (7).

Denote with S^*_{k-CS} the set of all k-coalition stable states and with S^*_{SS} the set of all sequentially-stable states. Define the *neighborhood* $\mathcal{N}_S \subset \mathbf{S}_n$ of $G \in \mathbf{S}_n$ as the set of networks which are at most one link or vertex different than G

$$\mathcal{N}_S = \left\{G' : G' = (g_{ij}, G_{-ij}), i \neq j\right\} \bigcup \left\{G' : G' = (a_i, G_{-i})\right\}$$

With these definitions in mind, the following proposition holds (its proof can be found in the appendix).

Proposition 3 *With assumption 1:*

[19]It can be shown that the k-player game has always a Nash equilibrium in pure strategies. In the case of multiple equilibria, by default, the one with the highest potential is chosen.

1. *For any* $1 < k < n$*, there exists at least one k-coalition stable state. Moreover the set of all k-coalition stable states is characterized by:*

$$S^*_{k-CS} = \left\{ G \in \mathbf{S}_n : G^k_n \in \arg\max_{G^k_n} \mathcal{P}, G^k_n = \{s_{i_r}\}^k_{r=1} \right\} \tag{10}$$

2. *There exists at least one sequentially-stable state. Moreover the set of all sequentially-stable states is characterized by:*

$$S^*_{SS} = \left\{ G : G = \arg\max_{G' \in \mathcal{N}_S} \mathcal{P}(G', X) \right\}$$

3. *The three equilibrium concepts are linked in the following way:*

$$S^{**}_{NE} \subseteq S^*_{(n-1)-CS} \subseteq ... \subseteq S^*_{2-CS} \subseteq S^*_{SS}$$

where $S^{**}_{NE} \in S^*_{NE}$ *is the Nash equilibrium which maximizes* $\mathcal{P}$*.*

The first two parts of proposition 3 present a tool to analyze the different notions of equilibria, which is useful whenever I introduce dynamics of the system (see proposition 4 and theorem 10). Their proofs consist of relatively straightforward algebraic manipulations, which rely on the existence of potential.

The conclusion of part 3 of proposition 3 gives us a clear idea about the relation between different equilibria. Intuitively, as k increases, the coordination requirements embedded in the definition of a k-coalition stable state become stronger and hence the number of equilibria shrinks. The conclusion of part 3, in fact does, not need the hypothesis of assumption 1. It is relatively straightforward to establish it from the definitions above.

3.4 Best Response Dynamics

Let us return to the dynamics of the behavioral model developed earlier and demonstrate that it naturally arrives at a stable state. This is the main result of this section, formalized in proposition 4. It requires to make the following, fairly mild, assumption 2. Later on, theorem 9 and theorem 10 extend it to alternative decision protocols.

Assumption 2 *For any configuration of the network and the attributes of the population, any meeting is possible:*

$$\Pr\left(\mu_t = (i, j) | G, X\right) > 0$$

for all $i, j \in I$*,* $G \in \mathbf{S}_n$ *and* $X \in \mathbf{X}_n$*.*

A desirable property of any equilibrium concept for a dynamical system is to be an attractor; that is with the time the system to approach such a state. Moreover one would expect an equilibrium to be a stable state: once reached, the system to remain there forever. A moment of reflection reveals that the set of sequentially-stable states obey the last property. The first property is somewhat more technical and may not appeal to intuitions.

Proposition 4 *With assumptions 1 and 2 the set of all equilibria of the network formation game coincides with the set of all sequentially-stable states. In particular:*

1. *Any $G \in S^*_{SS}$ is absorbing*
2. *Independently of the initial condition (distribution)*

$$\Pr\left(\lim_{t\to\infty} G_t \in S^*_{SS}\right) = 1$$

The first part of proposition 4 is implied immediately from the definition of S^*_{SS}. In this settings, the second part follows elegantly from the observation that the potential function $\{\mathcal{P}_t\}$ is a sub-martingale.[20] It then follows as a corollary that the network formation game always has an equilibrium.

3.5 A k-player Stage Game Dynamics

This section relaxes the seemingly restrictive assumption of the best response dynamics that a single player can revise either her action or a given link to a more general settings. In particular, suppose that at every period a coalition of size k is drawn to play the following game: (k is fixed throughout)

1. The set of players is $\{i_1, i_2, ..., i_k\}$
2. The strategy space is given by $S_{i_r} = \{0,1\}^k$, for $r \in \{1, ..., k\}$, with typical element

   $$s_{i_r} = \left(a_{i_r}, \{g_{i_r i_{r'}}\}_{r,r'=1}^{k}\right)$$

   with $r \neq r'$.
3. The payoffs are given in equation (7), which is rewritten here for convenience:

$$\begin{aligned} u_i(G) \quad = \quad & a_i v(X_i) + a_i \sum_j a_j h(X_i, X_j) + \sum_j g_{ij} g_{ji} a_i a_j \psi + \\ & \sum_j g_{ij} w(X_i, X_j) + \sum_j g_{ij} g_{ji} m(X_i, X_j) \end{aligned}$$

To complete the description of the setup, a proper extension of the meeting process is needed. Suppose that in every period a coalition of fixed size k is chosen ($1 < k \leq n$). The meeting process is then:

$$\Pr\left(\mu_t = \{i_1, i_2, ..., i_k\} | G_{t-1}, X\right) = \mu_{\{i_1, i_2, ..., i_k\}}\left(G_{t-1}, X\right) \tag{11}$$

i.e. it maps the state of the network to a probability distribution over the outcomes:

$$\mu_t : \mathbf{S}_n \times \mathbf{X}_n \longrightarrow \Delta\left(\wp_k(I)\right)$$

[20] Formal proof is available upon request.

Here $\wp_k(I)$ stands for the set of all subsets of I of size k i.e. $\wp_k(I) = \{I' \subseteq I : |I| = k\}$.

For the dynamics of the k-player stage game, a similar to proposition 4 result holds i.e. with time the system naturally arrives at a stable state. Consider the following adaptation to assumption 2.

Assumption 3 *For any configuration of the network and the attributes of the population, any meeting is possible:*

$$\Pr\left(\mu_t = I' | G, X\right) > 0$$

for all $I' \in \wp_k(I)$, $G \in \mathbf{S}_n$ and $X \in \mathbf{X}_n$.

Before we proceed with the statement of a formal theorem we need to specify what equilibrium will be played in case the stage game has multiple equilibria. In such a case, if nothing else is explicitly assumed, suppose that the equilibrium with the highest potential is chosen. Recall that k is kept fixed over time.

Proposition 5 *With assumptions 1 and 3 the network formation has the following properties:*

1. *Every k-player stage game has a pure strategy Nash equilibrium.*
2. *Any $G \in S^*_{k-CS}$ is absorbing.*
3. *Independently of the initial condition (distribution)*

$$\Pr\left(\lim_{t\to\infty} G_t \in S^*_{k-CS}\right) = 1$$

While the dynamic perspective introduced above addresses the concern raised by Kandori et al. (1993), it exhibits some undesirable properties. Importantly it cannot explain how the system developed after an equilibrium is reached. This imposes some technical issues when bringing the model to the data. These concerns are addressed in the next section.

4 An Estimable Model

To reflect more realistically the fact that there are unobserved factors, which affect players considerations in the model, this section introduces a random preference shocks to the utilities from peer selection and agents actions. Relevant to the context of application could be, for example, appearance characteristics, contextual effects, mood, emotional predisposition, etc. The incorporation of preference shocks also eliminates absorbing states of the network and allows us to obtain a non degenerate likelihood. Later on in this section, I explore particular distributional assumptions which guarantee a closed-form expression for the likelihood.

Assumption 4 *Suppose that the utilities in (4) and (5) contain a random preference shock. More specifically for $k = 1, 2$ let*

$$\overline{U}_{i,k} = U_{i,k} + \epsilon_{i,k}$$

with $\epsilon_{i,k} \sim$ i.i.d. across time, links, and nodes. Moreover suppose that ϵ has unbounded support on $\mathbb{R}$.

Observation 1 *The matching process* $\{\mu_t\}_{t=1}^{\infty}$ *and the sequence of optimal choices, in terms of friends selection and individual actions, induce a Markov chain of network configurations on* $\mathcal{G}_n$*.*

Now I present the first result of this section. Since its proof is notationally dense it is developed in details in the appendix. First, I lay out a formal statement of a theorem and then discuss its implications.

Theorem 6 *With assumptions 1, 2, and 4 the Markov chain generated by the network formation game has the following properties:*

1. *There exists a unique stationary distribution* $\pi \in \Delta(\mathcal{G}_n)$*. That is if the chain has unconditional distribution* π *at period* t*, it has the same distribution for any* $t' > t$*.*

2. *Independently of the initial condition*

$$\lim_{t \to \infty} \Pr(G_t = G) = \pi(G)$$

   that is the G_t *converges in distribution to* $G \sim \pi$*.*

3. *For any function* $f : \mathcal{G}_n \longrightarrow \mathbb{R}$ *the ergodic theorem holds*

$$\frac{1}{T} \sum_{t=0}^{T} f(G_t) \xrightarrow{a.s} \bar{f}$$

   where $\overline{f} = \int f(G)\, d\pi$*.*

The conclusion of theorem 6 is especially relevant for the empirical application of the model in a number of directions. First, it demonstrates that equilibrium play induces a unique stationary distribution over all network states and hence all different notions of equilibria receive positive probability. Realistically, a network state will receive positive probability although it may not be an equilibrium in any sense. Note however, that in the vicinity of an equilibrium, the equilibrium will receive the highest probability as it maximizes the potential and as theorem 7 asserts the likelihood is a monotone function of the potential.

The second part of the conclusion of theorem 6 asserts that the stationary distribution π can be recovered from a large number of sequential choices of the individuals and the corresponding network configurations. Specifically, draws from the stationary distribution can be generated via simulations of the individual's optimal choices.

To understand the third part of theorem 6, note that in the special case when f is the indicator function of state G it just states that the proportion of time spent at each state converges almost surely to the inverse of the equilibrium probability of that state (also the expected return time). This last result allows us to simulate moments from the likelihood function given a sequence of optimal decisions (and whence network configurations) consistent with the model.

The following assumptions allow me to obtain a closed form expression for the likelihood.[21]

[21] See Appendix B for details on the Gumbel distribution.

Assumption 5 *Suppose that the preference shock in the utility defined in assumption 4 is distributed Gumbel*($\underline{\mu}, \underline{\beta}$).

Assumption 6 *Suppose that for the meeting probability* μ*:*

1. $\Pr(\mu_t = (i,j))$ *is a symmetric function in* X_i *and* X_j*, and does not depend on the relationship state between* i *and* j*. In particular for all* $G \in \mathcal{G}_n$ *and* $X \in \mathbf{X}$

$$\mu_{i,j}(G,X) = \mu_{j,i}(G,X)$$

2. $\Pr(\mu_t = (i,i))$ *does not depend on* a_i*.*

Theorem 7 *Under assumptions 1-6, the stationary distribution* π*, from theorem 6, is characterized up to a constant by:*

$$\pi(G,X) \propto \exp\left(\frac{\mathcal{P}(G,X)}{\underline{\beta}}\right) \tag{12}$$

There are number of advantages that a closed from expression of the likelihood function presents. First, one can explore a transparent argument for the identification of the model parameters. It is clear that, given the variation in the data of individual choices $\{a_i\}_{i=1}^n$, friendships $\{g_{ij}\}_{i,j=1}^n$ and attributes $\{X_i\}_{i=1}^n$, functional forms for v, w, m, h will be identified as long as the different parameters induce different likelihood of the data. Second, a closed form expression for the likelihood is handy if the empirical researcher intends to explore Monte Carlo techniques to simulate the likelihood and estimate the model (say via Bayesian estimation). Last, but not least, theorem 7 allows one to analyze how restrictive are the different behavioral assumptions on the decision protocol. The last point becomes clearer below.

Define a state S as a *long-run* equilibrium of the underlying theoretical model if for any sequence of vanishing preference shocks, the stationary distribution π places a positive probability on S. The following theorem provides important probabilistic ranking of the network states. Its conclusion gives a new context of the equilibrium selection result for the risk dominant equilibrium of Kandori et al. (1993).

Theorem 8 *Suppose assumptions 1-6 hold:*

1. *A state* $S \in S^*_{SS}$ *is sequentially stable i.f.f. it receives the highest probability in its neighborhood* $\mathcal{N}_S$*.*

2. *The most likely network configuration (and the one where the network spends most of its time in) is a Nash equilibrium of the one-shot game, which do not need to be the Pareto dominant Nash equilibrium.*

3. *There is a unique long-run equilibrium of the underlying theoretical model which is given by* S^{**}_{NE}*.*

5 Extensions

The sequential decision protocol is standard in the network literature relying on stochastic best response dynamics. I show that in this section that, it is not particularly restrictive. Indeed under alternative decision protocols, the statements of propositions 1 and 4, as well as versions of theorems 6 and 7 continue to hold. In what follows, I discuss two alternatives.

5.1 Multiple Simultaneous Decisions

Suppose as before that in every period an agent i meets a random peer say j. However, let i be able not only to revise her "friendship" status with j but also her action a_i. That is agent i chooses $g_{ij}, a_i \in \{0, 1\}$ jointly to maximize her utility.

$$\begin{aligned} U_i\left(a_i, g_{ij} | G_{-i,-ij}, X\right) &= U_{1,i}\left(a_i | (g_{ij}, G_{-i,-ij}), X\right) + \\ &\quad U_{2,i}\left(g_{ij} | (a_i, G_{-i,-ij}), X\right) - a_i a_j g_{ij} g_{ji} \phi_{ij} \end{aligned} \tag{13}$$

where U_1 and U_2 are defined as before in (4) and (5). The intuition for the last (correction) term in the joint utility U_i is that now the peer effect goes in two directions: from actions to links and reversely. Because the decision now is simultaneous and because this peer effect enters separately both in U_1 and U_2, such a correction is necessary.

With the modification of utilities given in (13) and an additional assumption on the meeting process, the statements of all previous results continue to hold.

Assumption 7 *In addition to assumption 6, suppose that* $\Pr(\mu_t = (i, j))$ *does not depend on* a_i.

Theorem 9 *With utility function given by (13) the statements of proposition 1, proposition 4, theorem 6, and theorem 7 hold mutatis mutandis. In particular the two models are observationally equivalent (have the same likelihood).*

For proof see the appendix.

5.2 A (Random k) k-player Stage Game

This section presents, what appears to be, a very unrestrictive decision protocol where every period a (random k) k-players participate in the stage game. The main conclusion is surprising in its generality. This protocol induces the same stationary distribution over the network states as the one from sequential protocol. It follows then that these two models are observationally equivalent - have the same likelihood.

Consider the dynamics of the network induced by a model where every period k randomly chosen agents $\{i_1, i_2, ..., i_k\}$ play the one shot game[22] where the strategy space is $S_{i_r} = \{0, 1\}^k$,

[22] It can be seen shown that this game has at least one pure-strategy Nash equilibrium. In the case of multiple equilibria, the agents select the one with the highest potential.

for $r \in \{1, ..., k\}$, with typical element

$$s_{i_r} = \left(a_{i_r}, \{g_{i_r i_{r'}}\}_{r,r'=1}^{k} \right)$$

with $r \neq r'$. The payoffs are given by

$$\overline{u}_{i_r}(G) = u_{i_r}(G) + \epsilon_G \tag{14}$$

for u from equation (7) and ϵ is i.i.d. extreme value type I across time and network states.

To complete the description of the setup, a proper extension of the meeting process is needed. Suppose that in every period a coalition of *random size and composition* is chosen. That is every period first k is drawn and then a random collection of k players is selected from the population $(1 < k \leq n)$. The meeting process is then:

$$\Pr\left(\mu_t = \{i_1, i_2, ..., i_k\} | G_{t-1}, X\right) = \mu_{\{i_1, i_2, ..., i_k\}}\left(G_{t-1}, X\right) \tag{15}$$

i.e. it maps the state of the network to a probability distribution over the outcomes:

$$\mu_t : \mathbf{S}_n \times \mathbf{X}_n \longrightarrow \Delta\left(\wp(I)\right)$$

Here $\wp(I)$ stands for the power set of I.

Surprisingly, with proper modification of the assumptions on the meeting process, the main results continue to hold. In essence, the proof of theorem 10 follows those of theorems 6 and 7 (It can be found in the appendix).

Theorem 10 *The dynamics of the network induced by Nash equilibrium play of the (random k) k-player stage game obeys theorem 6 and theorem 7 mutatis mutandis. In particular the (random k) k-player model and the sequential model are observationally equivalent.*

Theorem 9 and theorem 10 demonstrate that the proposed framework is robust to different techniques for capturing the strategic aspect of individuals' behavior i.e. alternative modeling techniques result in observationally equivalent models. This is important in that it demonstrates that the model predictions do not hinge on ad hoc assumptions.

6 Discussion

This section relates the outcome of the estimable model to the structure of the underlying theoretical model from section 3. The link is discussed from the perspective of the notion of quantile response equilibrium introduced by McKelvey and Palfrey (1995) and the notion of correlated equilibrium introduced by Aumann (1974).

The notion of a quantile response equilibrium (QRE) is based on a fixed point of the quantile-response functions (QRFs) very much like Nash equilibrium is a fixed point of the best response functions. The QRF of player i is a smoothed best response function, where the strictly rational choice of player i (i.e. the best response) is replaced by an approximately rational response. A (regular) quantile-response function satisfies the following axioms:

1. Interiority: every strategy, in i's strategy space, receives strictly positive probability.

2. Continuity: the probability of player i choosing pure strategy s is a continuously differentiable function of the i's expected payoff of choosing s.

3. Responsiveness: the first derivative of the above probability is strictly positive for all players on the support of the expected payoffs.

4. Monotonicity: strategies with higher expected payoff receive higher probability.

The existence of a (regular) quantile response equilibrium of a finite-player finite-strategy space normal form game, trivially follows from the Brower's fixed point theorem. Any such equilibrium induces a probability distribution π^{QRE} over $S = S_1 \times S_2 \times ... \times S_n$ where S_i is the set of pure strategies of player i. Note that π^{QRE} is the Cartesian product of the equilibrium quantile responses and whence inherits their properties - i.e. the conditional distributions satisfy the axioms above.

The network formation protocol developed in section 4 and extended above induces a unique probability distribution (likelihood) π over the set of all possible network states $\mathbf{S}_n$. Interestingly π bears some similarities with the axioms of QRF inherited in π^{QRE}. However, there are important differences. I discuss each in turn.

Proposition 11 *The conditional distribution of player i's choices (i.e.* $a_i, \{g_{ij}\}_{j \neq i}$*) on the choices of the rest of the players -* $\pi_{(i)|(-i)} \in \Delta(\{0, 1\}^n)$ *induced by the optimal play in the network formation model, satisfies all properties of a (regular) quantile response function - interiority, continuity, responsiveness, and monotonicity. However, a QRE can not induce* π*.*

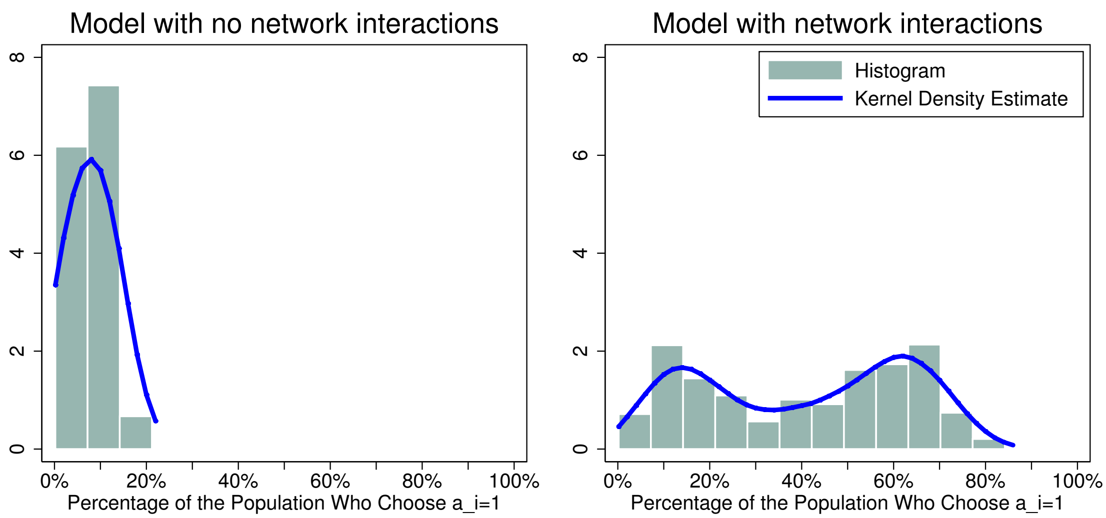
The first part follows trivially from the expression for π, given by theorem 7. Since the discrete network formation game is a game with potential, all payoffs can be represented by a single function of the network state - the potential $\mathcal{P}$. Importantly, the potential ranks the network states consistently with individual preferences and, at the same time, is linked to the probability of observing a given state. Thus it is not surprising that π exhibits the intuitive properties of a QRF.

For the second part, note that π in theorem 7 is not necessarily the product of the marginal probability distributions over each players choices $\{a_i, \{g_{ij}\}_{j \neq i}\}$. Thus a QRE can not induce π. Moreover, the outcome of the play bears similarities with the notion of correlated equilibrium of Aumann (1974) as it *cannot* be induced by a mixed strategy profile of the underlying n player one shot network formation game.

7 Concluding remarks

This paper develops a framework to study discrete games where the network structure plays a role in affecting agents decisions and determining the outcome. The proposed theoretical model accommodates both decisions regarding network links and decisions regarding individual actions.

Figure 2: Equilibrium Action Prevalences



Note. Simulation of the sequential model with a sequence of 10^9 *network configurations. Both panels plot a histogram and a kernel density estimate of the proportion of population who choose* $a_i = 1$ *(action prevalence) in given period. For the simulation on the left side, the parameters governing the endogenous and aggregate peer effects (*ϕ *and* $h(.,.)$ *in (4)) are set to zero i.e. the network does not affect the choice of* a_i*.*

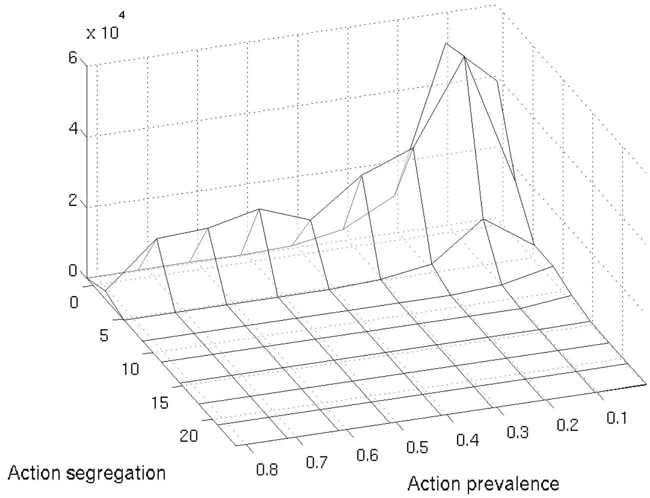
Alternative concepts of equilibrium network states are discussed and existence is established using the notion of potential function.

This paper proposes an estimable version of the theoretical model capable of handling multiplicity of equilibria. Indeed, as figure 2 suggests multiple equilibria are plausible in the structural model. Conveniently however, the behavioral assumption of the framework generate probabilities to all equilibria (see figure 2 and the note below). To guide the implementation, properties of a generic estimable specification are discussed. In addition, a closed-form expression for the likelihood is obtained, which is advantageous in that it readily presents an identification argument for model's parameters and facilitates the estimation.

I use the likelihood to argue that this framework is robust to different techniques for capturing the strategic aspect of individuals' behavior i.e. the sequential model of network dynamics is observationally equivalent to, what is termed as, (random k) k-player stage game dynamics. That is to say that the model where every period one individual plays a best response induces the same stationary distribution over the space of network configurations $\mathbf{S}_n$ as the model where every period a coalition of random size and composition acts according to a Nash play. This is important in that it demonstrates that the model predictions do not hinge on ad hoc assumptions. In addition, the sequential framework is computationally easier to estimate with appropriate data.

A substantial body of research has focused on how social interactions shape individual

Figure 3: Histogram action prevalence and segregation



Note. 3-D histogram of the action prevalence (the number of people taking action $a_i = 1$) and the level of segregation from a simulated sequence of networks (simulation of size 10^9 from the sequential model). Higher segregation levels are consistent with low action prevalence. Network configurations with hight action prevalences are less likely to occur and whenever they occur such networks feature lower segregation. Thus segregation can be thought of as containing the "contagion" of action choice $a_i = 1$.

behavior and more generally how the network affect agents choices. My approach is attractive for examining individual behaviors and evaluating feasible policy alternatives in situations, where the mechanism for intervention is likely to operate through altering aggregate (social) norms and, more generally, through the structure of the network (See figure 3 and the note bellow for an illustration[23]).

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[23] In figure 3, the coefficient of segregation is calculated as the number of links between nodes i and j for which $a_i(1 - a_j) = 1$ normalized by the expected number of links with this property. Note that the expected number of such links depends on the size of the population and the size of the pool of individuals who choose $a_i = 1$.

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A Proofs of the Main Results

Proof (Proposition 1) Recall that at period t individuals are assigned one of two choices - to choose their action $a_i \in \{0,1\}$ or to befriend a random acquaintance $g_{ij} \in \{0,1\}$. Then it suffices to verify two conditions.

Condition A. For any i, G_{-i} and X

$$\mathcal{P}(S', X) - \mathcal{P}(S, X) = U_1(a_i = 1|S_{-i}, X) - U_1(a_i = 0|S_{-i}, X) \tag{16}$$

where $S, S' \in \mathbf{S}_n$ are defined as $S = (a_i = 0, S_{-i})$ and $S' = (a_i = 1, S_{-i})$.

Condition B. For any $i \neq j$, S_{-ij} and X

$$\mathcal{P}(S', X) - \mathcal{P}(S, X) = U_2(g_{ij} = 1|S_{-ij}, X) - U_2(g_{ij} = 0|G_{-ij}, X) \tag{17}$$

where $S, S' \in \mathbf{S}_n$ are defined as $S = (g_{ij} = 0, S_{-ij})$ and $S' = (g_{ij} = 1, S_{-ij})$.

With the conditions of assumption 1 these are trivial to verify. Substitute for $\mathcal{P}(.,.)$ in the left-hand-side of condition (16):

$$\begin{aligned}
\mathcal{P}(S', X) - \mathcal{P}(S, X) &= v(X_i) + \frac{1}{2}\sum_{j\neq i} a_j g_{ij} g_{ji}(\phi_{ij} + \phi_{ji}) + \frac{1}{2}\sum_{j\neq i} a_j(h(X_i, X_j) + h(X_j, X_i)) \\
&= U_1(a_i = 1|S_{-i}, X)
\end{aligned}$$

The last equality is sufficient because of the normalization $U_1(a_i = 0|S_{-i}, X)$. The verification of condition (17) proceeds similarly. ∎

Proof (Proposition 3) Start with part 3 first. For $S \in S^*_{NE}$ no player has incentive to deviate in the unrestricted strategy space. But then no player will have incentive to deviate in the game of $(n-1)$ players as well. Thus $S^*_{NE} \subseteq S^*_{(n-1)-CS}$. The argument is completed similarly.

Since S^*_{NE} is non empty - the maximum of the potential is a Nash equilibrium, then the other equilibrium sets are non-empty by what was established above. This is the first parts of 1 and 2.

Finally the characterizations of the set S^*_{k-CS} follows from the fact that all agents play best responses and moreover in case of multiple equilibrium in the k-players game, the one which maximizes the potential is chosen. Then it follows that the strategies of any k players maximize the potential conditional the strategies of the rest $n-k$ players.

Proof (Proposition 5) First observe that the k-player stage game is a game with potential $\mathcal{P}$. Indeed since the n player stage game is a game with potential then naturally the restriction to k player has a potential as well. Hence the strategies that maximize the potential constitute a Nash equilibrium in pure strategy (this argument is similar to the one in Rosenthal (1973)).

That any k-stable state is absorbing follows from the definition of S^*_{k-CS}. The final part of proposition 5 follows from the following claim:

Claim 1 *The sequence of the values of the potential $\mathcal{P}_t$ induced by the outcome of the k-player stage games is a submartingale. i.e.*

$$E[\mathcal{P}_{t+1}|S_t] \geq \mathcal{P}_t \tag{18}$$

So that $\{\mathcal{P}_t\}$ converges almost surely. Since the network is of finite size it follows that $\{\mathcal{P}_t\}$ is constant for large t. Because of assumption 3 this can happen only if $S_t \in S^*_{k-CS}$.

Proof (Observation 1) First note that $\Pr(S_{t+1}|S_t) > 0$ only if $S_{t+1} \in \mathcal{N}_{S_t}$; that is there is a positive transition probability only to near-by networks. For those transitions, the probability equals the product of the probabilities of two events - the matching probability that appropriate edge/vertex is going to be chosen and the probability that the respective action is going to be optimal. Both of these probabilities are functions of S_t only (vs $\{S_{t'}\}_{t'=1}^{t}$) and whence the chain exhibits the Markov property. ∎

Proof (Theorem 6) The *existence* of equilibrium (aka stationary, invariant) distribution over the state space $\mathbf{S}_n$ is trivial algebraic property as the (finite dimensional) transition matrix has a row eigen vector with eigen value of 1. The rest of the properties can be established as a consequence of the following claim.

Claim 2 *The Markov chain of network configurations $(S_t)_{t\geq 0}$ induced by the matching process $\{\mu_t\}$ and the optimal choice of individuals is:*

1. *irreducible*
2. *positive recurrent*
3. *aperiodic*

Recall that the state space is *finite*. Fix the current state to G. Since the preference errors have unbounded support any state S' in the neighborhood of S - $\mathcal{N}_S$, communicates with S. Since all states can communicate with each other the Markov chain has a single class i.e. it is *irreducible*. The class is trivially closed and whence *recurrent* (Every finite closed class is recurrent); that is the probability that every state will be revisited infinitely often equals one. The chain is trivially aperiodic: there exists N such that $\Pr(S_{t+n} = S'|S_t = S) > 0$ for all $n > N$ and $S, S' \in \mathbf{S}_n$. This completes the proof of the claim.

The stationary distribution π is unique as the chain is irreducible and (positive) recurrent. The last two properties are related to convergence to the stationary distribution. Since in addition the chain is aperiodic

$$\Pr(S_t = S) \longrightarrow \pi(S)$$

that is S_t converges to the stationary distribution π no matter where the chain starts. Finally since the chain is irreducible and positive recurrent for any (trivially bounded on $\mathbf{S}_n$) $f : \mathbf{S}_n \longrightarrow \mathbb{R}$ we have the conclusion of the ergodic theorem:

$$\frac{1}{n}\sum_{t=0}^{T} f(S_t) \xrightarrow{a.s.} \overline{f}$$

where $\overline{f} = \int f(S)\, d\pi$. ■

Proof (Theorem 7) The proof uses the so called detailed balance property: if π and $\{S\}$ are in detailed balance, then π is invariant for $\{S\}$. Let:

$$\begin{aligned} S &= (a_i = 0, S_{-i}, X) \\ S' &= (a_i = 1, S_{-i}, X) \end{aligned}$$

then the detailed balance for S, S' is satisfied:

$$\begin{aligned} \Pr(S'|S)\pi(S) &= \mu(i,i)\Pr(U_1(a_i=1)>0)\frac{\exp\{\mathcal{P}(S)/\underline{\beta}\}}{H} \\ &= \mu(i,i)\frac{\exp\{U_1(a_i=1)/\underline{\beta}\}}{\exp\{U_1(a_i=0)/\underline{\beta}\}+\exp\{U_1(a_i=1)/\underline{\beta}\}}\frac{\exp\{\mathcal{P}(S)/\underline{\beta}\}}{H} \\ &= \mu(i,i)\frac{\exp\{\mathcal{P}(S')/\underline{\beta}\}}{\exp\{\mathcal{P}(S')/\underline{\beta}\}+\exp\{\mathcal{P}(S)/\underline{\beta}\}}\frac{\exp\{\mathcal{P}(S)/\underline{\beta}\}}{H} \\ &= \mu(i,i)\Pr(U_1(a_i=1)<0)\frac{\exp\{\mathcal{P}(S')/\underline{\beta}\}}{H} \\ &= \Pr(S|S')\pi(S') \end{aligned}$$

where the constant H equals $\sum_{S\in\mathbf{S}_n}\exp\{\mathcal{P}(S)/\underline{\beta}\}$. Note that, in order to avoid clutter, part of the argument of U_1 is omitted. The rest of the cases can be shown similarly. ■

Proof (Theorem 9) The argument hinges on the fact that under the two decision protocols the game has the same *potential*. To see this define:

$$\begin{aligned} S &= (a_i = 0, g_{ij} = 0, S_{-i,-ij}, X) \\ S' &= (a_i = 1, g_{ij} = 0, S_{-i,-ij}, X) \\ S'' &= (a_i = 0, g_{ij} = 1, S_{-i,-ij}, X) \\ S''' &= (a_i = 1, g_{ij} = 1, S_{-i,-ij}, X) \end{aligned}$$

To demonstrate that $P(.,.)$ is a potential function it suffices to show that

$$\mathcal{P}(S',X) - P(S,X) = U(1,0|S_{-i,-ij},X) - U(0,0|S_{-i,-ij},X) \tag{19}$$

$$\mathcal{P}(S'',X) - P(S,X) = U(0,1|S_{-i,-ij},X) - U(0,0|S_{-i,-ij},X) \tag{20}$$

$$\mathcal{P}(S''',X) - P(S'',X) = U(1,1|S_{-i,-ij},X) - U(0,1|S_{-i,-ij},X) \tag{21}$$

Indeed upon substitution of the definition of U from (13) the above derivations follow trivially. ■

Proof (Theorem 10) The existence and uniqueness of stationary distribution over $\mathbf{S}_n$ of the (random k) k player dynamics follows the proof of theorem 6. In essence the argument hinges the assumptions of unbounded support for the error term and the assumptions on the meeting process, which now includes the size and the composition of the playing coalition. What remains to be shown is that the (random k) k player dynamics induces the same stationary distribution π.

Indeed consider the detailed balance property for $S, S' \in \mathbf{S}_n$ and let S and S' differ by the play of the coalition $I_k = \{i_1, i_2, ..., i_k\}$. Then the probability of moving from state S and S' can be decomposed as the probability of coalition $I'_{k'} = \{i'_1, i'_2, ..., i'_{k'}\}$ of size $k' \geq k$ to meet such that:

$$\{i'_1, i'_2, ..., i'_{k'}\} \supseteq \{i_1, i_2, ..., i_k\} \tag{22}$$

and the probability of coalition $I'_{k'}$ taking the respective decisions consistent with S'. Formally

$$\begin{aligned}
\Pr(S'|S)\pi(S) &= \sum_{I'_{k'} \supseteq I_k} \mu(I'_{k'}) \Pr(S'|S, I'_{k'}) \frac{\exp\{\mathcal{P}(S)\}}{H} \\
&= \sum_{I'_{k'} \supseteq I_k} \mu(I'_{k'}) \frac{\exp\{\mathcal{P}(S')\}}{H'} \frac{\exp\{P(S)\}}{H} \\
&= \sum_{I'_{k'} \supseteq I_k} \mu(I'_{k'}) \Pr(S|S', I_{k'}) \frac{\exp\{\mathcal{P}(S')\}}{H} \\
&= \Pr(S|S')\pi(S')
\end{aligned}$$

where the constant $H' = \sum_{S'' \in \mathbf{S}_{(n-k')}(I'_{k'})}$ and as before $H = \sum_{S \in \mathbf{S}_n} \exp\{\mathcal{P}(G)\}$. ∎

B Gumbel distribution

The Gumbel distribution, also known as log-Weibull, double exponential, and type I extreme value, is a two parameter family of continuous probability distributions. Its probability density function $f(.|\mu, \beta)$ and cumulative distribution function $F(.|\mu, \beta)$ are:

$$\begin{aligned}
f(x|\mu, \beta) &= \frac{1}{\beta} \exp\{-z - \exp\{-z\}\} \\
F(x|\mu, \beta) &= \exp\{-\exp\{-z\}\}
\end{aligned}$$

where $z = \frac{x-\mu}{\beta}$. Its mode, median, and mean are μ, $\mu - \beta \ln(\ln 2)$, and $\mu + \gamma\beta$ respectively ($\gamma \approx 0.5772$ is the Euler-Mascheroni constant); Its variance is $V(X) = \frac{\beta^2\pi^2}{6}$. The following property of the Gumbel distribution makes it particularly attractive one in the analysis of qualitative response (aka quantal, categorical, or discrete) models.

Lemma 12 *Let $V_i \in \mathbb{R}$ for $i = 1, ..., n$ and ϵ_i are i.i.d. Gumbel(μ, β) random variables. For $X_i = V_i + \epsilon_i$,*

$$\Pr(X_i = \max_j X_j) = \frac{\exp\{V_i/\beta\}}{\sum_{j=1}^n \exp\{V_j/\beta\}} \tag{23}$$

The proof proceeds through a sequence of algebraic steps. First note that $\Pr(X_i = \max_j X_j) = \Pr(\epsilon_j < V_i - V_j + \epsilon_i, \quad \forall j \neq i) = \mathbb{E}_{\vec{\epsilon}}\left[\prod_{j\neq i} \chi_{\epsilon_j < V_i - V_j + \epsilon_i}(\epsilon_j)\right]$. Since ϵ_i are independent:

$$
\begin{aligned}
\Pr(X_i = \max_j X_j) &= \mathbb{E}_{\epsilon_i}\left[\mathbb{E}_{\vec{\epsilon}|\epsilon_i}\left(\prod_{j\neq i} \chi_{\{\epsilon_j < V_i - V_j + \epsilon_i\}}(\epsilon_j)\right)\right] \\
&= \int \left[\prod_{j\neq i} F(\epsilon_j < V_i - V_j + \epsilon_i)\right] f(\epsilon_i) d\epsilon_i \\
&= \int \left[\prod_{j\neq i} \exp\left\{-\exp\left\{-\frac{V_i - V_j + \epsilon_i - \mu}{\beta}\right\}\right\}\right] \frac{1}{\beta}\exp\left\{-\frac{\epsilon_i - \mu}{\beta} - \exp\left\{-\frac{\epsilon_i - \mu}{\beta}\right\}\right\} d\epsilon_i \\
&= \int \exp\left\{-\sum_{j\neq i} \exp\left\{-\frac{V_i - V_j}{\beta} - z\right\}\right\} \exp\left\{-z - \exp\left\{-z\right\}\right\} dz \\
&= \int \exp\left\{-\exp\left\{-z\right\} \underbrace{\sum_{j\neq i} \exp\left\{-\frac{V_i - V_j}{\beta}\right\}}_{v}\right\} \exp\left\{-z\right\} \exp\left\{-\exp\left\{-z\right\}\right\} dz \\
&= \int_{-\infty}^{0} \exp\left\{z' v\right\} \exp\left\{z'\right\} dz' \\
&= \frac{\exp\{V_i/\beta\}}{\sum_j \exp\{V_j/\beta\}}
\end{aligned}
$$

where $z = \frac{\epsilon_i - \mu}{\beta}$, $z' = -\exp\{-z\}$ and $\chi_{\{A\}}(.)$ is the characteristic function of the set A.