Diagnostic Bubbles

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1 Introduction

The financial crisis of 2007-2008 has revived academic interest in price bubbles. Robert Shiller (2015) created a famous graph of home prices in the United States over the course of a century, which shows prices being relatively stable during most of the 20th century, then doubling over the ten year period after 1996, only to collapse afterwards. But there is also growing evidence of speculation such as buying for resale in the housing market (DeFusco, Nathanson, and Zwick 2018, Mian and Sufi 2018) as well as of increasing leverage of both homeowners and financial institutions related to rapid home price appreciation. The collapse of the housing bubble is at the heart of every major narrative of the financial crisis and the Great Recession because it entailed massive losses for homeowners, holders of mortgage backed securities, and financial institutions (Mian and Sufi 2014). Recent evidence suggests that the U.S. experience in 2008 is not unique and that many leverage expansions and subsequent crises are tied to bubbles in housing and other markets (Jorda, Schularick and Taylor 2015).

Despite the revival of academic interest, asset price bubbles remain controversial in finance. Economic historians tend to take the existence of price bubbles to be self-evident (e.g., Kindleberger 1978), but getting to the bottom of how they work, and even whether they exist, has proved challenging. Early economic research has focused on rational price bubbles that do not violate (some definitions of) market efficiency (e.g., Blanchard and Watson 1982, Tirole 1982), but these models are not consistent with the available evidence on prices (Giglio, Maggiori and Stroebel 2016). More importantly, they are not consistent with the striking evidence of excessively optimistic expectations in bubble episodes (Shiller 2007, Greenwood and Shleifer 2014, Gennaioli and Shleifer 2018). Fama (2013) raised the critical empirical issue of whether price bubbles in fact exist in the sense of predictability of future negative returns after prices have

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increased substantially. Interestingly, since the pathbreaking paper of Smith, Suchanek and Williams (1988), predictably negative returns are commonly found in laboratory experiments even when markets have finite horizons. Greenwood, Shleifer, and You (2018) address Fama’s challenge in actual stock return data around the world, and find that although the predictability of returns is indeed difficult to establish with confidence based only on past returns, other indicators of over-pricing do help predict poor returns going forward. Theoretical research on bubbles tried to understand some of their main features, such as the roles of innovation (Pastor and Veronesi 2006), speculation (Scheinkman and Xiong 2003), price extrapolation (Barberis et al. 2018), and the supply factors (Glaeser 2008, Glaeser and Nathanson 2015). Despite substantial advances, the structure of asset price bubbles remains an open problem.

In this paper, we try to make progress on understanding the structure of price bubbles using a psychologically founded model of belief formation. We start with Kindleberger’s (1978) view of bubbles as consisting of three stages. The first is displacement, meaning a period of good fundamental news, often tied to an economic innovation, that lead to rapid asset price increases. The second and crucial period is the acceleration of price growth, as price increases themselves feed buying and further price increases, and prices reach levels substantially above fundamental values. The third period is the price collapse, as traders recognize that prices cannot grow to the sky, and sell the assets. We do not in this paper examine leverage and other factors that link the collapses of bubbles to financial fragility and economic recessions; our goal is to link the structure of expectations to Kindleberger’s three stages. We try to understand the connection between our formulation of expectations and the central features of bubbles, such as speculation, price extrapolation, and eventual price collapses. We show that our model of expectations allows for a unified account of many of the central facts about both prices and beliefs.

We examine the model of expectations introduced in our earlier work, which we called Diagnostic Expectations (see Bordalo, Gennaioli, and Shleifer, hereafter BGS 2018, Bordalo, Gennaioli, La Porta and Shleifer, hereafter BGLS 2018, and Bordalo, Gennaioli, Ma, and Shleifer, hereafter BGMS 2018). This model relates beliefs to Kahneman and Tversky’s (1972) representativeness heuristic. Representativeness refers to the notion that, in forming probabilistic assessments, decision makers put too much weight on outcomes that are likely not in absolute terms, but rather relative to some reference or baseline level. For
example, many people significantly over-estimate the probability that a person’s hair is red when told that the person is Irish. The share of red-haired Irish, at 10%, is a small minority, but red hair is much more common among the Irish than among other Europeans, let alone in the world as a whole. The over-estimation of the prevalence of representative types distorts beliefs and accounts for many systematic errors in probabilistic judgments documented experimentally by Kahneman and Tversky and others (see Gennaioli and Shleifer 2010). It also delivers a theory of stereotypes that has been shown to be consistent with both field and experimental evidence, including gender stereotypes in assessments of ability (Bordalo, Coffman, Gennaioli, and Shleifer, herafter BCGS 2016, 2018), racial stereotypes in decisions about bail (Arnold, Dobbie and Yang 2018), and popular beliefs about immigrants (Alesina, Miano, and Stantcheva 2018).

The model of representativeness applies directly to understanding belief formation in dynamic contexts, which we refer to as diagnostic expectations. The central question is how much decision makers update their beliefs in response to new information. Under Rational Expectations, the answer to this question is given by the Bayes Rule. Under Diagnostic Expectations, instead, decision makers update their beliefs too far (relative to the rational benchmark) in the direction of the states of the world whose objective likelihood has increased the most in light of recent news. Thus, after good news, right-tail outcomes become representative and are overweighed in expectations, while left-tail outcomes become non-representative and are neglected. Diagnostic Expectations yield a psychologically founded formulation of over-reaction to news, which seems to be consistent with a good deal of evidence, including expectations of credit spreads (BGS 2018), earnings growth (BGLS 2018), and macroeconomic variables (BGMS 2018).

We incorporate diagnostic expectations into a standard finite horizon model of a market for one asset, in which there is a continuum of investors receiving noisy private information every period about the termination value of that asset. Because they receive different noisy signals, traders hold heterogeneous beliefs, which are observed in expectations data (BGMS 2018). Heterogeneity allows us to study trading volume, which is an important feature of bubbles (Scheinkman and Xiong 2003). In our benchmark model, traders have long horizons in that they maximize their final period utility. We examine the model starting with Kindleberger’s displacement, whereby agents receive consistently good news about the value of the asset, and then ask what happens to the price. Under Rational Expectations, the price converges smoothly
from the lower prior up to the fundamental value. Whether or not there is learning from prices in addition to
private information, the price never overshoots, and hence there are no price bubbles in equilibrium. This
benchmark model then allows us to examine the consequences of diagnostic expectations.

We conduct our analysis in three steps. First, we consider the simplest case in which traders use
only their private information, and do not learn from prices. Second, we consider the same model, but allow
traders to learn from prices. Finally, we change the model to consider short horizon traders whose objective
incorporates not the final value of the asset, but rather next period price. We can thus examine speculation –
buying for resale – which is a central feature of many narratives of the bubbles, including Kindleberger’s.

The three cases help us shed light on the anatomy of price bubbles. Under the basic specification in
which traders rely only on their private information, prices exhibit three phases. Initially, they underreact,
remaining below the fundamental value of the asset. This occurs because early on there is a lot of uncertainty
about the value of the new asset, so traders discount their signals heavily, despite representativeness. As
good fundamental signals keep coming, however, uncertainty shrinks. This effect makes the right tail of
possible outcomes highly representative and hence inflated: a large number of traders now think that the
asset is truly spectacular, despite the fact that such assets are rare (i.e., they believe there are too many
Googles). The price overshoots, rising above fundamental values. In the third phase, the bubble runs out of
steam: the marginal value of recent news declines, which cools of traders’ excess optimism. The price
slowly converges to the rational benchmark. This model can account for some over-pricing, but does not
generate price acceleration that is so central to most narratives of the bubble. This is due to the fact that in
this model price inflation is tightly linked to over-optimism about fundamentals.

Allowing for learning from prices addresses some of these shortcomings. As good news come in and
prices rise, diagnostic traders act more aggressively on their private signals, which makes prices more
informative than under the rational benchmark. As a consequence, traders react even more aggressively to
price signals, which quickly swamp the less informative private ones, and cause prices to accelerate upwards.
This formulation delivers a bubble in the form not only of prices exceeding fundamental values in
equilibrium, but also – critically – of price acceleration as the bubble develops. In fact, our crucial finding is
that, in this model, it looks like investors are extrapolating price trends, even though they are not. Price
extrapolation reflects an overreaction to recent price increases, which leads traders to upgrade (too much) their expectations of fundamental value, and thus of the future price. As a consequence, price extrapolation arises only when learning from prices is powerful, at the later stages of the bubble. This pattern is different from models of adaptive expectations, such as Barberis et al. (2018), which yield a constant price extrapolation. At the same time, although the model with diagnostic learning from prices fits better the Kindleberger narrative than the initial formulation, it does not deliver wild prices because equilibrium prices are still tethered to eventual liquidation values.

In the final version of the model, we introduce speculation, whereby traders optimize relative to their beliefs of the next period price rather than the eventual liquidation value of the asset. This modification has dramatic consequences. In this market, even without learning from prices, traders can drive prices extremely high, because in the bubble they exaggerate the over-optimism of traders in the following period. The mechanism is similar to the beauty contest logic: each trader correctly believes that other traders will overreact to their future signals, but his expectations of the latter are too inflated. As before, after seeing several pieces of good news, a trader today is more confident that the asset is a Google than is warranted in reality. Crucially, however, he also believes that future traders will over-react to this belief, and so will be even more confident the asset is a Google. Overreaction compounds as each trader looks ahead, and the expectation of reselling the asset to very bullish traders strongly inflates the price today. Eventually, as the terminal date approaches, the opportunities for resale become scarcer and the bubble collapses.

The conceptual lesson we learn is that speculation, along with diagnostic expectations, is a central ingredient of price bubbles, at least in terms of the three Kindleberger phases. When combined with learning from prices, speculation creates a disconnect between expectations of fundamentals and expectations of price increases. Even a modicum of fundamental overheating, as measured by a small distortion coefficient $\theta$, can compound into strong price extrapolation and large price dislocations.

We develop the paper in line with the three model formulations just described. Section 2 introduces the basic setting, and examines the case where traders learn from private signals under diagnostic expectations. Sections 3 and 4 introduce learning from prices and speculation, respectively. Section 5 concludes.
2. The basic setting: learning from good shocks.

We consider a setting where traders learn about the value of a new asset, and trade it, over a finite set of periods $t = 0, ..., T$. The asset yields a payoff $V$, which is drawn from a normal distribution with mean 0 and variance $\sigma_V^2$ at $t = 0$ but is only revealed at the terminal date $T$. In line with Kindelberger’s (1978) description of a positive displacement as the trigger of bubble episodes, we focus on the case of a valuable innovation, $V > 0$. In each period $t$, each trader $i$ (in measure one) receives a private signal $s_{it} = V + \epsilon_{it}$ of the asset’s value. Noise $\epsilon_{it}$ is i.i.d. across traders and over time, and normally distributed with mean zero and variance $\sigma_\epsilon^2$. Because the new asset is valuable, $V > 0$, traders are repeatedly exposed to good news, in that signals are on average positive relative to their priors, generating the initial displacement. Moreover, the assumption of dispersed information generates variation in expectations and helps account for trading, both of which are important features of bubble episodes.

We study learning and price dynamics under three increasingly rich specifications of this basic setting. We assume that traders are risk averse with CARA utility $u(c) = -e^{-\gamma c}$. In this section, we further assume that traders have no speculative motive and do not learn from prices. That is, they buy the asset with the expectations of keeping it – so they only care about its fundamental value $V$ – and they estimate $V$ solely on the basis of their private signals. In Section 3 we augment this basic model by assuming that traders also learn from prices. In Section 4 we additionally assume that traders have a speculative motive. The deterministic nature of our setting (in the sense that the aggregate signal at any time $t$ is equal to $V$) places strong constraints on the price paths under rational expectations in all these specifications, in particular ensuring that the average price path monotonically increases towards $V$. None of the models we consider can create bubble-like behavior under rationality. In contrast, even in the stripped-down model of this section, where traders learn only from their private signals, diagnostic expectations generate boom-bust price dynamics. Adding learning from prices and speculative trading then illustrates the key role of diagnostic expectations in driving not only overshooting but also several other features of bubble-like episodes.

In the absence of a speculative motive, it is straightforward to see that the evolution of prices is fully determined by the evolution of consensus beliefs about the fundamental value $V$. To see this, suppose trader

\footnote{Without loss of generality we abstract away from time discounting.}
Trader $i$’s demand $D_{it}$ of the asset maximizes the mean-variance objective function:

$$D_{it} = \max_{D_{it}} \left[ \mathbb{E}_{it}(V) - p_t \right] D_{it} - \frac{\gamma}{2} \text{Var}_t(V) D_{it}^2,$$

where $\gamma$ captures risk aversion. Trader $i$’s demand $D_{it}$ is then given by:

$$D_{it} = \frac{\mathbb{E}_{it}(V) - p_t}{\gamma \text{Var}_t(V)}, \quad (1)$$

Intuitively, demand increases in the difference between the trader’s valuation and the market price.

For simplicity we assume here that the supply of the asset is zero. By equating the aggregate demand obtained from Equation (1) to this zero supply, we see that the equilibrium price at time $t$ satisfies:

$$p_t = \int \mathbb{E}_{it}(V) di. \quad (2)$$

Simply put, the price of the asset is equal to the consensus estimate of $V$. Whether diagnostic expectations can generate bubble-like dynamics therefore depends on their implication for the dynamics of consensus beliefs. We introduce diagnostic expectations in Section 2.1 and derive the consensus beliefs in Section 2.2.

2.1 Diagnostic Expectations

Diagnostic Expectations (BGS 2018) describe a mechanism whereby individual beliefs overweigh representative (although possibly unlikely) future outcomes. According to Kahneman and Tversky (1972), the reliance on representativeness as a proxy for likelihood is a central feature of probabilistic judgments. As they put it, an outcome “is representative of a class if it’s very diagnostic”, that is, if its “relative frequency is much higher in that class than in the relevant reference class” (Kahneman and Tversky 1983). Gennaioli and Shleifer (2010) formalize the representativeness heuristic and show that it generates a number of well-established fallacies of probabilistic assessments. This model has been subsequently shown to shed light on

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3 It is clear that beliefs are normal in the setting described above under rational expectations. In the next section we show this also holds under diagnostic expectations.
the formation of stereotypes (BCGS 2016). For example, the stereotype of an Irish person as having red hair arises not because having red hair is likely (only 10% of the Irish people do) but because it is much more likely among the Irish than among other national groups (roughly 1% of the European population have red hair). Under representativeness, unlikely but distinctive traits of otherwise similar groups have disproportional weight in beliefs.

Diagnostic Expectations (BGS 2016, BGLS 2018, BGMS 2018) adapt this approach to dynamic, stochastic processes, by defining a future outcome to be more representative when it has become objectively more likely in light of recent data, even if it still unlikely in absolute terms. Suppose an investor updates his beliefs about the distribution of a variable of interest $X$ in light of new information at time $t$. In our setting, $X$ is the asset’s payoff $V$ (or its future price); in previous work, $X$ was firms’ earnings growth (BGLS) or macroeconomic variables (BGMS). Denote by $f(X_{t+1}|G_s)$ the true distribution conditional on information at time $s$, denoted $G_s$. A realization $X_{t+1}$ becomes more representative at time $t$ if $X_{t+1}$ scores higher on representativeness, defined by:

$$\frac{f(X_{t+1}|G_t)}{f(X_{t+1}|G_{t-k})}.$$  

Put differently, $X_{t+1}$ is more representative if it has become more likely in light of recent data $G_t$ relative to the reference information $G_{t-k}$ held $k$ periods before. The reference distribution $f(X_{t+1}|G_{t-k})$ captures what the investor perceives, at time $t$, as the normal state of affairs. Lag $k$ captures the speed with which the perception of normal conditions adjust. For $k = 1$, they adjust continuously to the most recent past. In reality, perception of normal conditions can be more sluggish: in our model, traders may hold the perspective that the norm for $V$ is zero for several periods, until they accumulate sufficient information. In fact, an estimation of diagnostic expectations model based on actual earnings of companies and analyst expectations of long term earnings growth suggests $k$ is around 3 years (BGLS 2018). Here, we take $k$ as given (typically at the estimate above), and explore its role in setting a time-scale in the dynamics of bubbles.

Representativeness implies that beliefs overreact to news by overweighing outcomes that have become more likely relative to the norm, and underweighting outcomes that have become less likely. The diagnostic distribution is then defined as (see Appendix A for details):

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\[ f^\theta (X_{t+1}| G_t) = f(X_{t+1}| G_t) \left[ \frac{f(X_{t+1}|G_t)}{f(X_{t+1}|G_{t-k})} \right]^\theta Z_t, \]

where \( \theta \geq 0 \) is a parameter capturing the strength of the representativeness distortion and \( Z_t \) is a normalizing constant.\(^4\) As shown in BGS (2018), if the variable \( X \) is normally distributed (and jointly normal across time), the diagnostic distribution above is also normal. In particular, if \( E_s(X_{t+1}) \) is the rational expectation at time \( s \), then the diagnostic expectation at time \( t \) is:

\[
E^\theta_t (X_{t+1}) = E_{t-k} (X_{t+1}) + (1 + \theta) \left[ E_t (X_{t+1}) - E_{t-k} (X_{t+1}) \right],
\]

(3)

Several features stand out in Equation (3). Diagnostic expectations are forward looking in the sense that they update in the correct direction, \( E_t (X_{t+1}) - E_{t-k} (X_{t+1}) \), but -- crucially -- too much. Expectations overreact to information by exaggerating the difference between current conditions \( E_t (X_{t+1}) \) and normal conditions \( E_{t-k} (X_{t+1}) \) by a factor of \( (1 + \theta) \). To see why, note that after good news, the likelihood of outcomes \( X_{t+1} \) in the right tail of the distribution increases; while such outcomes are still unlikely, they become hugely representative after good news because their prior probability was so low (in fact the representativeness ratio is monotonically increasing in \( X_{t+1} \)). Diagnostic expectations therefore put disproportionate weight on the right tail, and conversely, neglect the risk in the left tail. For normal distributions, this reweighting results in an excessive rightward shift of beliefs. Because distortions are driven by actual news, diagnostic expectations (like stereotypes) contain a kernel of truth. Finally, the model nests rational expectations as the case of zero representativeness distortions, \( \theta = 0 \).

BGLS show that diagnostic expectations obtained from Equation (3) account well for the link between listed firms’ performance and equity analysts’ expectations of their future earnings growth, as well as, crucially, for the link between expectations and the predictability of their stock returns. They also

\(^4\) Formally, in a Markov process with density \( f(X_{t+1}|X_t) \), after a particular current state \( \tilde{X}_t \) is realized and in light of the past expectation for the current state \( E_{t-k}(X_t) \), the distorted distribution is equal to:

\[ f^\theta(X_{t+1}|X_t) = f(X_{t+1}|X_t) \left[ \frac{f(X_{t+1}|\tilde{X}_t)}{f(X_{t+1}|E_{t-k}(X_t))} \right]^\theta Z_t, \]

where both the target distribution \( f(X_{t+1}|\tilde{X}_t) \) and reference distribution \( f(X_{t+1}|E_{t-k}(X_t)) \) have the same variance. Roughly speaking, beliefs overweighs states that have become more likely in light of the surprise \( \tilde{X}_t - E_{t-k}(X_t) \) relative to \( k \) periods ago.
estimate the model and find, with quarterly data, that \( \theta \approx 1 \) and \( k \approx 3 \) years. A similar value of \( \theta \) has been estimated using expectations of credit spreads by BGS, and using macroeconomic forecasts by BGMS.

We now apply this formalism to our setting where traders learn about an asset’s value based exclusively on their own private signals. In this case, the diagnostic expectation \( E_{it}^\theta(V) \) held by trader \( i \) at time \( t \) is easily characterized. Recall that each period, the trader observes a private signal \( s_{it} = V + \epsilon_{it} \) of the asset’s true value. Having observed the string of signals \((s_{i1}, \ldots, s_{it})\), the rational belief about \( V \) is normal with mean \( E_{it}(V) = \pi_t \bar{s}_{it} \), where \( \bar{s}_{it} \equiv \frac{\sum_{r=1}^{t} s_{ir}}{t} \) is the average signal observed by trader \( i \) while \( \pi_t \) is the signal to noise ratio: 

\[
\pi_t \equiv \frac{t/\sigma^2}{t/\sigma^2 + 1/\sigma_V^2},
\]

which is common across all traders. Note that the conditional distribution of \( V \) depends only on private signals because we are ruling out learning from prices, which we introduce later.

Under diagnostic expectations, the information provided by the cumulative signal \( \bar{s}_{it} \) is assessed relative to the information available \( k \) periods before, which is summarized by the prior assessment of value \( E_{i,t-k}(V) \). During early periods, when \( t \leq k \), the norm is anchored to the prior beliefs, so that \( E_{i,t-k}(V) = 0 \). Eventually, however, as \( t > k \), early signals are progressively incorporated into the norm, which becomes \( E_{i,t-k}(V) = \pi_{t-k} \bar{s}_{it-k} \). In line with Equation (3) we then obtain:

**Lemma 1** Given signal \( \bar{s}_{it} \), the diagnostic beliefs held about \( V \) by trader \( i \) are normally distributed with the rational variance \( \text{Var}_t(V) = (1 - \pi_t)/\sigma^2_V \) and the distorted mean:

\[
E_{it}^\theta(V) \equiv \begin{cases} 
(1 + \theta)\pi_t \bar{s}_{it} & \text{for } t \leq k \\
\pi_t \bar{s}_{it} + \theta(\pi_{t-k} \bar{s}_{it-k} - \pi_{t-k} \bar{s}_{it-k}) & \text{for } t > k 
\end{cases}.
\]

The trader overreacts to the last \( k \) signals he has observed. Early on, for \( t \leq k \), he over-reacts to all signals \( \bar{s}_{it} \), inflating the rational assessment by the factor \((1 + \theta)\). Because the new asset is valuable, \( V > 0 \), each trader on average receives good news each period during this stage, i.e. \( \bar{s}_{it} > 0 \). As described above, this increases the objective probability that \( V \) is in the right tail of its prior distribution. As a result, high
values of $V$ become representative and low values of $V$ unrepresentative. This causes the trader to exaggerate the likelihood of high values of the fundamental, which induces an excessive shift to the right in his mean belief. Subsequently, as $t > k$, over-reaction to early signals fades. Early signals have become part of the norm, and the trader only overreacts to the most recent signals, as captured by the term $(\pi_t \bar{s}_{lt} - \pi_{t-k} \bar{s}_{lt-k})$.

While the basic structure of overreaction to noisy signals and adaptation follows BGLS (2018), the current setting differs from in two important ways: first, traders hold heterogeneous beliefs about $V$. Second, the precision of beliefs increases over time, as reflected by the fact that $\pi_t$ increases in $t$. These differences play a key role in reproducing the features of a price bubble.

### 2.2 Consensus Beliefs and Prices

To find equilibrium prices it is enough to aggregate the individual beliefs in Equation (4) into the consensus measure of Equation (2). Using the fact that the signal observed by the average trader is an unbiased estimate of fundamentals, $\int s_{it} di = V$, the equilibrium price is uniquely determined and is given by:

$$ p_t = \int \mathbb{E}_t^\theta(V) \, di = \begin{cases} (1 + \theta)\pi_t V & \text{for} \quad t \leq k \\ [\pi_t + \theta(\pi_t - \pi_{t-k})]V & \text{for} \quad t > k \end{cases}. $$

Consider first the rational benchmark, with $\theta = 0$. The equilibrium price satisfies $p_t = \pi_t V$ for all $t$, and thus monotonically converges to the true fundamental $V$ from below. Traders rationally discount their signals in light of noise $\sigma_i^2$, and have no access to information held by other traders. Despite the fact that the aggregate signal equals $V$, traders on average initially hold low beliefs about $V$ because the gain $\pi_t$ is low. Over time, each trader’s signal becomes more accurate, the gain $\pi_t$ increases, all traders converge to the true value, and price converges to fundamental. Through this inefficient aggregation of information, both consensus beliefs and prices exhibit upward momentum (Woodford 2003) but not overshooting relative to fundamentals.
Under diagnostic expectations, on the other hand, the price path exhibits the following boom bust dynamics:

**Proposition 1** If $\theta \in \left( \frac{1}{k} \frac{\sigma_2^2}{\sigma_0^2}, \frac{\sigma_2^2}{\sigma_0^2} \right)$, the equilibrium price path $p_t$ exhibits three phases:

1) Delayed over-reaction: $p_t$ starts below $V$, then increases gradually to its peak $p_k = (1 + \theta)\pi_k V$ at $t = k$.

2) Bust: $p_t$ drops at $t = k + 1$ and reaches its trough $p_t^* < V$ at $t^* \equiv k \cdot \max \left[ 1, \left( 1 - \frac{\theta}{\sqrt{1+\theta}} \right)^{-1} - \theta^* \right]$.

3) Recovery: $p_t$ gradually recovers for $t > t^*$, asymptotically converging to the fundamental $V$.

Consider each of the three phases in turn. At very early stages, the price under-reacts to the fundamental displacement, $p_t < V$. While individual traders overreact relative to the rational benchmark, provided individual overreaction is not too large, $\theta < \frac{\sigma_2^2}{\sigma_0^2}$, they still discount their private signals relative to the full information case. This ensures the underreaction of consensus expectations about $V$, and of prices, relative to the unbiased aggregate signal. The idea that aggregate underreaction survives in the presence of moderate individual overreaction to noisy signals was explored in BGMS (2018) in the context of macroeconomic forecasts.

As good signals keep accumulating, the noise perceived by individual traders diminishes, and their gain $\pi_t$ increases, to the point that $(1 + \theta)\pi_t > 1$. Now, not only individual traders, but also the consensus belief over-reacts, and the price overshoots the fundamental $p_t = (1 + \theta)\pi_t V > V$. This happens because the average trader has received sufficient good news. Relative to his prior of zero, these news make very representative the possibility that the innovation is truly great. Even for moderate overreaction, the average trader then puts such a large weight on the right tail of the distribution of $V$ that his belief overshoots even his own observed signal (which equals $V$)! This stands in stark contrast not only to the rational benchmark but also to any model of mis-specified learning in which beliefs are a convex combination of priors and new information. This distinctive feature reflects the fact that diagnostic expectations generate disproportional and asymmetric weight on tail events: if traders focus on the right tail, and neglect the left tail, then in some

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5 This overshooting happens for $t < k$ provided $\theta > \frac{1}{k} \frac{\sigma_2^2}{\sigma_0^2}$. For infinite $k$, overshooting occurs eventually for any positive $\theta$. 

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sense “the sky is the limit”: sufficiently many good signals about $V$ bring to mind stratospheric values, each fast growing firm is believed to be a new Google, trees are believed to grow to the sky. This result is very much in line with the standard narrative of bubbles, in which the initial displacement, which often comes from a major innovation, leads investors to believe in a “paradigm shift” that captures the most optimistic scenarios that could result from such an innovation.\textsuperscript{6}

This initial phase continues up to $k$, while incoming signals are compared to the reference value of $V = 0$. At $t = k + 1$, however, we enter the bust phase 2). Traders start perceiving the early news as normal, and focus instead on the new information contained in recent signals. However, because the fundamental $V$ is finite, recent signals have a smaller incremental value. As a result, the extent of over-reaction drops and the price collapses. This collapse continues while expectations adjust to the prior cumulative signal $\bar{s}_{it-k}$, which grows increasingly positive.

The bust here is due not to bad news, but to the declining pace at which good news arrive, which causes the bubble to run out of steam. This mechanism of waning over-reaction is one narrative of the collapse phase that arises naturally under diagnostic expectations, and it may help explain the evidence, such as that obtaining in the cross-section of stocks (BGLS 2018), that even positive news are a disappointment relative to what is hoped. We do not suggest that it is the only mechanism behind a bust; other factors including bad news (the housing bubble deflating from 2006 on), as well as the proximity of a terminal trading date (crucial in experimental findings), are surely significant. We consider the later mechanism in our model of speculation in Section 4.

At $t = t^*$ the price reaches its trough, which is close to the rational price $(1 + \theta)\pi_t V < V$. From this point onward, over-reaction to good news is negligible (because good news are minor), and the price gradually converges to $V$ from below.

Because traders observe independent signals, the model also generates time variation in heterogeneity of beliefs and thus in trading volume. Consider the dispersion of traders’ beliefs, as a proxy for trading volume, during the period when the bubble grows, $t \leq k$. The dispersion is given by

\textsuperscript{6} In Pastor and Veronesi (2009), a successful innovation is not initially overpriced, but instead becomes central enough to the economy that the risk associated with it becomes systematic, which in turn depresses prices.
\[ \text{Var}_t \left( \mathbb{E}_{t,t}^\theta (V) \right) = (1 + \theta) \sigma_e^2 t \left[ t + \frac{\sigma_e^2}{\theta} \right]^{-2}, \]

which increases over time until \( t < \sigma_e^2 / \sigma_v^2 \) and then declines.

Intuitively, traders’ beliefs initially diverge because of the noise \( \sigma_e^2 \) in their incoming signals is larger than the prior uncertainty. At \( t = \sigma_e^2 / \sigma_v^2 \) the rational valuation reaches \( V/2 \), and sufficiently many signals have been received that the noise falls below the prior uncertainty. As more signals are received, beliefs start converging again.

In sum, by introducing a degree of over-reaction \( \theta \) to recent news in an otherwise standard dispersed information model, diagnostic expectations can account for delayed over-reaction of prices to a fundamental displacement and for the subsequent bust as dramatic good news stop coming. As, at some point, the consensus expectations of value lie above the average observed signal. These dynamics cannot be obtained under rational updating. In fact, they cannot be obtained with any model of mis-specified beliefs such as overconfidence (Daniel, Hirshleifer and Subramanyam 1998) in which beliefs are a convex combination of priors and signals, which imply monotone price convergence to the fundamental \( V \).

Still, the price path generated by this model exhibits two evident limitations as the mechanism of the bubble. First, the price increase in the boom phase is concave, in the sense that price changes slow down over time. In fact, the rate of change of prices equal the rational rate for \( t < k \). Most bubble episodes, however, display convex price growth, a feature that some empirical methods exploit to detect bubbles in the data (Greenwood, Shleifer, and Yang 2018). Second, for realistic parameter values the extent of price inflation is limited. Using \( \theta = 1 \) as estimated using expectations of earnings growth (BGLS 2018), and also using expectations of macroeconomic time series (BGMS 2018), suggests that the price at the peak is bounded above by \( 2V \). In several historical episodes, such as the internet bubble, prices reached multiple times the plausible measures of fundamentals.

To address these issues, in the next section we make the model more realistic by relaxing the assumption that traders do not learn from prices. Learning from prices is an important feature of real world...
markets, whereby traders seek to extract from price changes the information that other traders have about fundamentals. Starting with Grossman (1976) and Grossman and Stiglitz (1980), learning from prices plays an important role in formal analyses of rational expectations equilibria in financial markets. The next section shows the workings of this mechanism under diagnostic expectations.

3. Diagnostic Learning from Prices

In our setting, if traders learn from prices and prices perfectly aggregate expectations, traders would learn the fundamental in one period. To make the learning problem interesting, we follow the literature on learning from prices and assume that that the supply $S_t$ of the asset is random, in particular it is i.i.d. normal with mean zero and variance $\sigma^2_S$. The classical justification for this assumption is the presence of noise or liquidity traders who demand/supply the assets for non-fundamental reasons (Black 1986, Grossman and Miller 1988, DeLong et al. 1990). The implication is that price is no longer fully revealing, as a high price today may due either to a low unobserved supply $S_t$ shock or to a high average private signal $V$.

In section 3.1 we solve for equilibrium prices and describe the key bubble-like properties of the average price path. Similar to the previous model, and as in Equation (2), the price reflects the consensus expectations about $V$. These expectations again exhibit overshooting bounded above by $(1 + \theta)V$. Now, however, traders learn about $V$ much faster than in the previous model, and in fact the price path can be convex in the early periods. Relatedly, learning from prices quickly outpaces learning from private information, so that after the first few signals the bubble is driven by information inferred from prices. In Section 3.2 we use simulations to explore the features of price extrapolation under diagnostic expectations which distinguish it from models of mechanical extrapolation.

3.1 Equilibrium Prices

We solve the model following the approach of rational expectations models (Grossman and Stiglitz 1980). We first conjecture that, at each time $t$, price is a linear function of the state variable of the economy, which include the fundamental $V$. Second, we assume that traders use this linear rule to make inferences
about $V$ in light of the current and past prices. Third, we determine at each time $t$ the coefficients of the pricing function that equilibrate demand and supply, so that the resulting rule yields the equilibrium price.

Denote by $\mathbb{E}(V|P_t)$ the rational expectation of $V$ based solely on the history of prices up to $t$, formally $P_t \equiv (p_1, \ldots, p_t)$. Then, our conjectured pricing rule takes the form:

$$p_t = a_{1t} + a_{2t}\mathbb{E}(V|P_t) + a_{3t}\mathbb{E}(V|P_{t-k}) + b_t \left( V - \frac{c_t}{b_t}S_t \right).$$

Equation (6) is reminiscent of rational expectations analyses. The current price reflects consensus expectations derived from all prices up to date, as well as the average private signal. Because diagnostic expectations combine current and lagged rational forecasts, the lagged forecast is also added as a state variable).

To solve for diagnostic expectations equilibrium we determine the coefficients $(a_{1t}, a_{2t}, a_{3t}, b_t, c_t)_{t \geq 1}$ that equate supply with demand when demand reflects traders’ diagnostic inferences from prices. We now sketch the solution, and leave a fuller account to the proof in Appendix A.

First, consider how traders learn in light of the conjectured pricing rule. Because the term $p_t - a_{1t} - a_{2t}\mathbb{E}(V|P_t) - a_{3t}\mathbb{E}(V|P_{t-k})$ as well as the coefficients $b_t$ and $c_t$ are known by all (in equilibrium), observing price $p_t$ effectively endows all traders with an unbiased public signal of $V$:

$$s_t^p = V - \left( \frac{c_t}{b_t} \right) S_t.$$

The mean of $s_t^p$ is the fundamental $V$ and its variance is $(c_t/b_t)^2 \sigma_S^2$. Price is more informative when it is more sensitive to the persistent fundamental than to the transient supply shock, namely when $c_t/b_t$ is lower.

Given that private and public signals $s_{it}$ and $s_{it}^p$ are normally distributed, conditional on a history of signals $(s_{i1}, \ldots, s_{it}, s_{i1}^p, \ldots, s_{it}^p)$ a rational trader’s beliefs about $V$ are normal, with mean and variance:

$$\mathbb{E}_{i,t}(V) = s_{i,t}z_t + \mathbb{E}(V|P_t)(1 - z_t),$$

$$Var_{i,t}(V) = \left[ \frac{t}{\sigma^2} + \frac{1}{\mathbb{E}(V|P_t)} \right]^{-1},$$

$$V_{ar}(V) = \left[ \frac{t}{\sigma^2} + \frac{1}{\mathbb{E}(V|P_t)} \right]^{-1},$$

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where \( z_t \in [0,1] \) is a known function of time and price coefficients and \( \text{Var}_t^P(V) \) is the variance of \( V \) solely based on the observation of prices.\(^8\)

The diagnostic expectation of \( V \) continues to follow Equation (4), which entails a consensus belief:

\[
\int \mathbb{E}_i^\theta(V)(V)di = \begin{cases} 
(1 + \theta)(1 - z_t)\mathbb{E}(V|P_t) + Vz_t & \text{for } t \leq k \\
(1 + \theta)(1 - z_t)\mathbb{E}(V|P_t) - \theta(1 - z_{t-k})\mathbb{E}(V|P_{t-k}) + [(1 + \theta)z_t - \theta z_{t-k}]V & \text{for } t > k
\end{cases}
\]

When traders learn from prices, diagnostic expectations introduce another distortion: exaggeration of the information revealed by prices. This exaggeration comes from the amplification \((1 + \theta)\) of the impact of the current price-based estimate \( \mathbb{E}(V|P_t) \), and from the reversal of past price inferences \( \mathbb{E}(V|P_{t-k}) \).

We start by considering the equilibrium for the boom phase of the bubble, \( t \leq k \).

**Proposition 2** In equilibrium with learning from prices and \( t \leq k \), the precision of the price signal increases over time as:

\[
\frac{b_t}{c_t} = \left( \frac{1 + \theta}{\gamma \sigma^2} \right) t, \tag{11}
\]

and the average equilibrium price path (for \( S_1 = S_2 = \ldots = S_t = 0 \)) is given by:

\[
p_t = (1 + \theta) \left[ \frac{1 + \theta}{\sigma^2} \frac{t(t+1)(2t+1)}{6} \right]^\frac{1}{2} \mathbb{E}(V|P_t). \tag{12}
\]

As in Proposition 1, for sufficiently large \( t \) price overshoots fundamentals, and is bounded above by \((1 + \theta)V\). But beyond Proposition 1, this result reveals two noteworthy properties. First, the price signal becomes more informative over time because transitory supply shocks tend to average out. Crucially, diagnostic expectations boost this effect (Equation 11). When \( \theta \) is high, investors are aggressive both in

---

\(^8\) \( \text{Var}_t^P(V) \) drops in the precision of the public signals observed up to \( t \), i.e.

\[
\text{Var}_t^P(V) = \left[ \frac{1}{\sigma^2} + \sum_{r=1}^{t} \left( \frac{b_r}{c_r} \right)^2 \frac{1}{\sigma^2} \right]^{-1},
\]

while the weight on private signals drops in the relative precision of prices, i.e.

\[
z_t = \frac{t}{\sigma^2} \left[ \frac{t}{\sigma^2} + \frac{1}{\text{Var}_t^P(V)} \right]^{-1}.
\]

See the appendix for details.
revising their beliefs and in trading on the basis of their private signals. Because these signals are informative about $V$, price informativeness increases. Unlike non-fundamental based behavioral biases, which disconnect prices from fundamentals, diagnostic expectations exaggerate this link, creating overreaction to the initial displacement. In turn, greater price informativeness implies that diagnostic traders over-react even more to price signals. By multiplying the distortion factors, prices are now inflated by the cubic power $(1 + \theta)^3$.

The second, and related, implication of Proposition 2 is that learning from prices progressively gains ground, and swamps private signals as prices rise up to the peak of the bubble. As shown in Equation (12) the precision of price signals grows with the cubic power of $t$, which eventually swamps the linear precision of private signals, $t/\sigma^2_\epsilon$. This mechanism has the following important implication.

**Proposition 3** With learning from prices, provided $\sigma^2_V$ is sufficiently small relative to $\sigma^2_S$ and $\sigma^2_\epsilon$, the price path is convex for small $t$.

Learning from prices can thus account for convex price growth. This occurs because eventually all traders, regardless of their private signals, aggressively infer fundamentals from the common signal of prices. Under rational expectations, learning from prices would also coordinate traders’ beliefs and generate convexity in prices, but would not be able to account for overvaluation. In contrast, under diagnostic expectations convex price growth eventually overshoots the fundamental, creating a bubble.

While the equilibrium with a linear pricing rule always exists for $t \leq k$, during the boom phase of the bubble, it may not exist for $t > k$. Preliminary analysis and simulations suggest that existence is guaranteed when the variance of fundamentals $\sigma^2_V$ is sufficiently small relative to the variance of supply shocks $\sigma^2_\epsilon$. When the equilibrium exists, it displays a boom bust pattern similar to the one described in Proposition 1.

### 3.2 Price Expectations

With the model with learning from prices, we can start to address an important feature of bubbles, namely the dynamics of expectations of future prices. While traders do not act on these expectations in the current model (holding the asset until the terminal period), understanding price expectations helps set the
stage for the model with speculation (Section 4). Importantly, by comparing such expectations to alternatives such as mechanical price extrapolation, we highlight distinct predictions of the model that can in principle be compared to empirical evidence.

Under mechanical extrapolation, traders automatically project past price increases into the future according to the updating rule:

$$\mathbb{E}_t(p_{t+1}) = p_t + \beta(p_{t-1} - p_t),$$

where $\beta > 0$ captures the fixed degree of price extrapolation. In our model, in contrast, traders watch prices in order to infer fundamentals. As a result, price extrapolation arises because high past prices signal high fundamentals and hence even higher future expected prices.

This logic suggests a testable difference between mechanical extrapolation models and price learning under diagnostic expectations. Under diagnostic expectations, the updating coefficient $\beta$ in Equation (13) should depend on the extent to which prices are informative. Modest price increases observed in the early stages of a bubble are not very informative about the magnitude of the fundamental, so the coefficient $\beta$ should be low. In contrast, a sustained price increase (as observed some time into the bubble) is a solid indicator of a strong fundamental, and should therefore be associated with a much higher over-reaction, as measured by the coefficient $\beta$ under diagnostic expectations.

We evaluate these ideas by simulating the model with price learning. We normalize the fundamental value $V$ to 1. To capture that displacement is a fairly rare event, we assume $\sigma_V = 0.5$. We set the dispersion of trader’s private signals at $\sigma_e = 12.5$, which is in the ballpark of what we estimated from the quarterly dispersion of professional forecasts (BGMS 2018). We set $\theta = 0.8$, in line with the quarterly estimates from macroeconomic and financial survey data. We take a time period to be a quarter, set the sluggishness of diagnostic beliefs at $k = 12$ (in line with the estimates from BGLS 2018) and we run the model for $T = 28$ periods, i.e. 7 years. Finally, we set the volatility of supply shocks at $\sigma_S = 0.3$.

Figure 1 below reports the actual price for the average path without supply shocks and the expected one period ahead price (averaged across traders) for two models: the diagnostic expectations model ($\theta = 0.8$) and the rational model ($\theta = 0$). To interpret the Figure, note that a trader’s expectation of tomorrow’s
price is equal to his expectation about tomorrow’s consensus estimate of $V$ minus a risk adjustment due to the random supply. The expected price is plotted at the forecast horizon, so the vertical difference between the price and its expectation captures the average forecast error.

Figure 1. Actual and Expected Prices

Under diagnostic expectations, the equilibrium price exhibits the typical boom bust pattern, where the boom is driven by overreaction to private signals and prices, while the bust is due to the reversal of expectations. In the rational model, by contrast, the price monotonically converges to $V$.

Consider now the dynamics of expected prices. Under diagnostic expectations, the expected price increases with the current price. During early periods, the expected price displays sluggishness. It does not increase as fast as the current price, so investors under-predict the price increase. This is due to the uncertainty faced initially by traders, which induces them to discount their private information when predicting the future consensus estimate of $V$. Subsequently, when the price crosses the fundamental $V$, the pattern reverses. Now expected prices grow fast, settling systematically above actual prices. This occurs because in this range traders overweight not only their information, but also how others will react to it in the future. As a result, the predicted consensus lies above the actual one.
These dynamics, which are key in the model with speculation in Section 4, are enhanced by learning from prices, which induces traders to extrapolate from past price increases, particularly in the convex portion when the price crosses the fundamental. To see the link between learning and price extrapolation, we run regression (13) using a time series of the model simulated using the parameters above. We produce 2000 such time series and plot in Figure 2 the histogram of estimated coefficients for both the diagnostic and the rational model.

**Figure 2. Model-Implied Extrapolation Coefficient**

The coefficient of price extrapolation implied by the model is positive, between .1 and .5. Even though our investors are entirely forward looking, they appear to mechanically extrapolate past prices. This is not the case under rational expectations, where the coefficient is negative. Intuitively, this is because a larger than usual price increase is likely due to a supply shock, which is then reversed the next period.

While diagnostic expectations entail a positive extrapolation coefficient on average across the entire bubble episode (as does mechanical extrapolation), the extrapolation coefficient is the highest when prices are most informative, which is the case close to the peak of the bubble. Figure 3 below reports the average estimates of the extrapolative coefficient $\beta$ in three buckets, capturing the growth, overshooting and collapse of the bubble (respectively, highlighted in blue, red, and green in the graph).
The results confirm our intuition: price extrapolation is strongest in the making of the bubble, when there is rapid learning (the first phase highlighted in blue). This occurs because prices are most informative (relative to the private signal) in that range, which induces diagnostic traders to update upward more aggressively after a price hike.\footnote{One could try to account for the patterns of Figure 4 with a model where mechanical extrapolation is nonlinear, and becomes stronger after larger price increases. This account however would not predict any relationship between forecasts of prices and forecasts of fundamentals, which go hand in hand under diagnostic expectations.}

In sum, diagnostic expectations offers a theory of price bubbles grounded in the psychology of representativeness. When combined with learning from prices, the theory can account for convex price dynamics and for extrapolation of past prices, which becomes particularly strong toward the peak of the bubble, when all investors take price increases as symptomatic of strong fundamentals.

Although learning from prices makes the model more realistic, it retains one shortcoming of the model of Section 1, namely that the maximum price in the current model is equal to \((1 + \theta)V\), which is roughly two times the fundamental for conventional values of \(\theta\). Often, however, prices appear to overshoot measures of fundamentals much more. We now show that such strong overvaluation can be attained by adding to our model another realistic and important ingredient: speculation.
4. Speculation

To introduce speculation, we assume that traders have short horizons in the sense that their objective function at each time $t$ is to resell the asset at time $t + 1$. The trading game lasts for $T$ rounds, and the traders holding the asset in the terminal date receive its fundamental value. In this setting, diagnostic expectations generate price paths with significantly larger overvaluation than in the previous models, followed by a price collapse as the terminal date approaches. This occurs not only because diagnostic speculators overreact to good fundamental news, but because they also expect to resell to overreacting buyers in the future, which drives the price today higher. As the terminal date approaches, the prospects for re-trading fade and the bubble bursts. As we will see, these dynamics are very different from those obtained with speculation under rationality.

As in previous Sections, traders hold mean-variance preferences (in particular, expectations are normal in the diagnostic equilibrium). Away from the terminal date, $t < T$, trader $i$ chooses demand $D_{it}$ to maximize $\left[ \mathbb{E}_i(p_{t+1}) - p_t \right] D_{it} - \frac{\gamma}{2} \text{Var}_i(p_{t+1}) D_{it}^2$, while his objective at time $T$ is fundamental-based as before. We can view this speculation motive as stemming from traders’ short horizons. We find:

\begin{align*}
D_{i,t} &= \frac{\mathbb{E}_{i,t}^θ(p_{t+1}) - p_t}{\gamma \text{Var}_i(p_{t+1})}, \quad \text{for } t = 1, \ldots, T - 1, \\
D_{i,T} &= \frac{\mathbb{E}_{i,T}^θ(V) - p_t}{\gamma \text{Var}_i(p_{t+1})}, \quad \text{for } t = T.
\end{align*}

Under speculation, demand increases in the expected capital gain $\mathbb{E}_{i,t}^θ(p_{t+1}) - p_t$ except in the last period $t = T$, in which traders buy the asset to hold it.

To find an equilibrium in the speculation model we again start from a linear pricing function, which is then used to compute price expectations $\mathbb{E}_{i,t}^θ(p_{t+1})$ for next period. In each period, the coefficients of the pricing function are pinned down by equating demand and supply. Because speculation is itself a source of price reversal as $T$ approaches, we simplify the analysis by assuming that the diagnostic reference is very sluggish, $k > T$, so that information about the asset’s value is always assessed compared to the prior $V = 0$.  

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We study the model in two steps. We first illustrate the key consequences of speculation in a simple model in which there is no learning from prices. In Section 4.2 we introduce such learning, which allows to compare this formulation to the one in Section 3.

### 4.1 Speculation without Learning from Prices

As in Section 2, absent learning from prices we assume that the supply of the asset is equal to zero. Aggregating the individual demand functions, the equilibrium conditions pin down the equilibrium prices:

\[ p_t = \int \mathbb{E}^\theta_{t,t}(p_{t+1}) \, di, \quad \text{for } t = 1, \ldots, T - 1, \tag{16} \]

\[ p_T = \int \mathbb{E}^\theta_{LT}(V) \, di. \tag{17} \]

In the final period \( T \), expectations of fundamental value \( \mathbb{E}^\theta_{LT}(V) \) are given by Equation (4) and lead to the price \( p_T = (1 + \theta)\pi_T V \), as in Section 2. Under the assumption \( \theta \in \left( \frac{1}{\sigma^2 \xi}, \frac{\sigma^2}{\alpha^2} \right) \) of Proposition 1, which we maintain, this price is above fundamental, \((1 + \theta)\pi_T > 1\).\(^{10}\)

Consider now the price at \( T - 1 \). By Equations (16) and (17) this is given by:

\[ p_{T-1} = \int \mathbb{E}^\theta_{j,T-1} \left( \int \mathbb{E}^\theta_{i,T}(V) \, di \right) dj = \int \int \mathbb{E}^\theta_{j,T-1} \left( \mathbb{E}^\theta_{i,T}(V) \right) dijdj. \]

In the presence of speculation, price depends on current expectations about next period’s beliefs. Trader \( j \) knows that in the following period the generic trader \( i \) will overreact to his own information, forming an estimate of fundamental value of \( \mathbb{E}^\theta_{LT}(V) = (1 + \theta)\pi_T s_{IT} \). In predicting this estimate, trader \( j \) is in turn swayed by representativeness, and overweighs his own information, so that \( \mathbb{E}^\theta_{j,T-1} \left( \mathbb{E}^\theta_{LT}(V) \right) = (1 + \theta)\pi_T \mathbb{E}^\theta_{j,T-1}(s_{IT}) = (1 + \theta)^2 \pi_T \pi_{T-1}s_{JT-1} \). Relative to his information set \( s_{j,T-1} \) at \( T - 1 \), trader \( j \) does

---

\(^{10}\) Recall that in this Section we shut down the adaptation of diagnostic expectations by setting \( k > T \).
two rounds of inference to predict beliefs in period $T$, but doing so compounds the diagnostic distortion. As every trader does the same, price at $T - 1$ is given by:

$$p_{T-1} = \int (1 + \theta)^2 \pi_T \pi_{T-1} \tilde{s}_{jt-1} d\tilde{j} = (1 + \theta)^2 \pi_T \pi_{T-1} V. \quad (18)$$

This is a key result to understand the role of speculation. If at $T - 1$ traders overshoot in the sense of $(1 + \theta)\pi_{T-1} > 1$, then Equation (18) implies that $p_{T-1} > p_T$. As the terminal period approaches, the prices falls. This is due to the fact that the trader at $T - 1$ exaggerates the over-optimism of traders at $T$: he correctly believes traders will overreact to their future information, but his expectations of the latter are too inflated. As a result, the price at $T - 1$ is too high. A similar argument goes through at $T - 2$: if in this period traders overreact to information, $(1 + \theta)\pi_{T-2} > 1$, they overreact to the prospect of selling the asset to a trader that at $T - 1$ in turn overreacts to the prospect of selling the asset at $T$. Hence, $p_{T-2} > p_{T-1} > p_T$.

In fact, the key statistic for price change at $t$, namely $(1 + \theta)\pi_{t-1}$, combines two forces: overreaction to signals, $\theta > 1$, and inferential discounting of signals, $\pi_{t-1} < 1$. For earlier periods, as the signal to noise ratio $\pi_{t-1}$ becomes smaller, traders still inflate future optimism but to a lesser extent. When discounting dominates, traders’ expectations $E_{j,t-1}(\tilde{s}_j)$ for the signals observed by future traders are still lower than their observed signals, $\tilde{s}_{jt-1}$, which implies a lower price at $t - 1$.

The full path of equilibrium prices is obtained by iterating Equation (18):

$$p_t = (1 + \theta)^{T-t+1} \left[ \prod_{r=t}^{T} \pi_r \right] V, \quad (19)$$

which implies the following result.

**Proposition 4** Define the geometric average of all signal to noise ratios $\bar{\pi} \equiv \left[ \prod_{r=1}^{T} \pi_r \right]^{\frac{1}{T}}$. Then, if $\theta \in \left( \frac{1 - \bar{\pi}}{1 - \frac{\sigma^2}{\rho^2}}, \frac{1 - \bar{\pi}}{\frac{1 - \pi}{\pi}} \right)$, where $\frac{1 - \bar{\pi}}{\frac{1 - \pi}{\pi}} < \frac{\sigma^2}{\rho^2}$, the speculative price dynamics exhibit two phases:

1. The price starts below fundamental, $p_1 < V$, and gradually increases above fundamentals and reaches its maximum at the smallest time $\hat{t}$ for which $(1 + \theta)\pi_{\hat{t}} > 1$.

2. From $t = \hat{t}$ onwards the price monotonically declines toward $p_T$.  

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Under diagnostic expectations, speculative dynamics can generate both the sluggish upward price adjustment typical of underreaction (provided $\theta$ is not too large), the price inflation relative to fundamentals typical of overreaction (provided $\theta$ is not too small), and the bust phase in which prices collapse, which here is driven by the reduction in the available rounds of reselling.

Because in the first period $(1 + \theta)^T \tilde{\pi} < 1$, individual traders underreact to the aggregate information because individual uncertainty about $V$ is still very large. As a consequence, these traders do not exaggerate much the optimism of next period buyers. This effect curtails the demand for the asset today, keeping its price low. As time goes buys, traders acquire more information and start using it more aggressively. They become more optimistic about the signals future buyers will get, more confident about future buyers’ over-optimism, and so the price starts increasing.

As traders gain confidence, and the possibility of multiple rounds of reselling dramatically boosts price, which overshoots $V$. The prospect of selling the asset to traders that strongly overreact to the fundamental displacement is now concrete, which causes the price to reach its maximum. The price starts declining as time approaches the terminal date $T$, because there are fewer and fewer rounds of trading and thus less scope for overreaction.

Under rationality, the dynamics of speculation are very different. Given $\theta = 0$, there is no countervailing force to the discounting of signals. Traders correctly expect that future traders have lower signals than they do, and that future traders in turn discount those signals, depressing the price today. Because traders receive information over time, price grows monotonically and approaches $V$ from below. It displays momentum but not overshooting nor reversal (Allen, Morris and Shin 2006).

One important implication of speculation is that it can greatly exacerbate the overshooting of fundamentals, relative to the benchmark model without speculation or learning from prices in Section 2. Equation (19) shows that speculation is the key fuel of bubbles under diagnostic expectations, and can cause strong price inflation even with moderate diagnostic distortions, $\theta \approx 1$. 

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4.2 Learning from Prices

We now enrich the speculative model with learning from prices. We saw already saw that learning from prices can create convex price increases and strengthen overreaction to information. This opens the possibility that inference from prices may contribute to speculative dynamics by inducing traders to expect strong price appreciation following good fundamental news.

To study learning from prices under speculation, we build on the analysis in Section 3. We start from the pricing rule of Equation (6), setting $V_{1i} = 0$ (which follows from our earlier analysis) as well as $V_{3i} = 0$, since for $k > T$ the lagged expectation is redundant. We also reintroduce supply shocks to prevent prices from being fully revealing. The key innovation is that traders now use the pricing rule also to forecast future prices, not only to learn about $V$. Equations (16) and (17) then imply the following market clearing conditions:

$$p_t = \int \mathbb{E}^\theta_{i,t}(p_{t+1}) di - \gamma Var_t(p_{t+1}) S_t, \quad \text{for } t = 1, \ldots, T - 1$$

$$p_T = \int \mathbb{E}^\theta_{i,T}(V) di - \gamma Var_T(V) S_T.$$

In line with Equation (9), the diagnostic expectation of value held by trader $i$ after observing the private signals $(s_{i1}, \ldots, s_{it}, s_{i1}^P, \ldots, s_{it}^P)$ is equal to:

$$\mathbb{E}^\theta_{i,t}(V) = (1 + \theta)[\bar{s}_{it} z_t + \mathbb{E}(V|P)(1 - z_t)],$$

where $z_t = \frac{t}{\sigma_e^2} \left[ \frac{1}{\nu_t^V} \right]$. The diagnostic expectation of price and its variance are then equal to:

$$\mathbb{E}^\theta_{i,t}(p_{t+1}) = (1 + \theta) \left[ a_{t+1} \mathbb{E}_{i,t}[E(V|P_{t+1})] + b_{t+1} \mathbb{E}_{i,t}(V) \right]$$

$$Var_t(p_{t+1}) = \left[ a_{t+1} \left( 1 - \frac{\nu_{t+1}^P}{\nu_t^P} \right) + b_{t+1} \right]^2 Var_t(s_{t+1}^P).$$

By plugging these expressions in the market clearing conditions we show the following result.

**Proposition 5** A diagnostic equilibrium at parameters $(\gamma, \theta, \sigma_e^2, \sigma_e^2, \sigma_V^2, T)$ is a set of coefficients $(a_{2t}, b_t, c_t)_{t=1, \ldots, T}$ and price-based variance $V_t^P$ satisfying the recursion for $t < T$:
\begin{align*}
  b_t &= (1 + \theta) \frac{t}{\sigma^2} \left[ (1 + \theta)^{T-t} - b_{t+1} \right] \left[ \frac{1}{V_{t+1}^p} \right]^{-1} \left( \frac{b_{t+1}}{c_{t+1}} \right) \frac{1}{\sigma^2} + b_{t+1} \\
  a_t &= (1 + \theta)^{T-t+1} - b_t, \\
  c_t &= \gamma \left[ (1 + \theta)^{T-t} - b_{t+1} \right] \left[ \frac{1}{V_{t+1}^p} \right]^{-1} \left( \frac{b_{t+1}}{c_{t+1}} \right) \frac{1}{\sigma^2} + b_{t+1} \\
  &\quad + \left( \frac{c_{t+1}}{b_{t+1}} \right)^2 \sigma^2, \\
  \frac{1}{V_t^p} &= \frac{1}{V_{t+1}^p} - \left( \frac{b_{t+1}}{c_{t+1}} \right)^2 \frac{1}{\sigma^2} \\

\end{align*}

with terminal conditions:

\begin{align*}
  b_T &= (1 + \theta) \frac{T}{\sigma^2} \left[ \frac{T}{\sigma^2} + \frac{1}{V_T^p (V)} \right]^{-1} \\
  c_T &= \gamma \left[ \frac{T}{\sigma^2} + \frac{1}{V_T^p (V)} \right]^{-1} \\
  \frac{1}{V_0^p} &= \frac{1}{\sigma^y} \\

\end{align*}

The above iteration specifies values at \( t \) as a function of those at \( t + 1 \), so that the recursion univocally identifies the equilibrium starting from the terminal condition. The equilibrium may not exist if (26) cannot be satisfied.

We simulate the model to explore its properties. To do so, we take the parameters \((\gamma, \theta, \sigma^2, \sigma^e, \sigma^y, T)\) used in Section 3, and find the terminal variance \(V_T^p\) (from which previous \(V_t^p\) are computed in equilibrium) such that condition (26) holds. Figure 5 below plots the average price path prevailing when the supply shock \(S_t\) is zero in all periods. For comparison, it also plots the corresponding paths under rational expectations.
Several features stand out. First, the price path displays the three phases described by Kindelberger (1978), namely a price growth phase, an acceleration and overshooting phase, and then a collapse. Second, the first phase is protracted, with prices being lower than in the rational case at first, before they accelerate. At this stage, inferences from prices actually slow down learning under diagnostic expectations. Third, when prices accelerate they do so strongly, and the extent of overpricing is much greater than that obtained in the previous model under the same parameters.

These features are driven by the interaction of speculation with learning from prices. The speculative motive adds uncertainty to the prices, but also allows traders to trade more aggressively on their beliefs. These forces play a differential role throughout the bubble episode. At first, traders have little information and prices are less informative about fundamentals. Traders thus heavily discount early price increases, which may be driven by a supply shock compounded by the willingness to speculate of other traders. Because all traders behave this way, prices (and beliefs about $V$) remain low. In contrast, once individual traders are confident that the innovation is in fact valuable, the interaction reverses sign. Learning from prices now fuels speculation, because current prices convey more information about other traders’ signals.
The price therefore skyrockets. As expected, muting learning from prices, by increasing the variance $\sigma^2$ of
the supply shock from 0.3 to 0.8, slows down learning even more, as shown in Figure 6 below.

Figure 6. Speculative bubbles and the informativeness of prices

![Figure 6](image)

The cautiousness that traders exhibit in seeking to tease out speculation from fundamentals in the
presence of supply shocks is made evident in the way they extrapolate prices. Figure 7 below runs
regression (13) for three different buckets of the speculative bubble, using the same parameters.

Figure 7. Price extrapolation during speculative bubbles

![Figure 7](image)

Price extrapolation increases as prices become more informative and learning takes off. This
parallels the findings of Figure 3, in which speculation is absent. In contrast to that case, however, the betas
of Figure 7 are negative throughout. This reflects the fact that traders expect much of the price movement (particularly at first) to be driven by supply shocks and thus to be reversed.

5. Conclusion

To be added
References.


