C The model with continuous types

In this appendix, we study a model with continuously many types \((\bar{\theta}, \theta)\). Recall that the distribution of these continuous types has a density function \(g(\theta)\). Before proceeding to the analysis, we provide a brief overview of the results in this appendix. With continuous types, the CEO’s misreporting behavior is characterized by two thresholds, \(\theta_1\) and \(\theta_2\), introduced below. Depending on the leniency parameter \(c\), the board determines a threshold level \(\theta_1\) such that it prefers to remove any CEO whose type is below this cutoff. Consequently, types lower than the threshold \(\theta_1\) must inflate their report in order to have a chance to remain as CEO. Indeed, an aggressive board sets \(\theta_1\) higher than the average ability \(E[\theta]\), meaning that even types above the average have an incentive to misreport if they are below the threshold. However, not all of the lower types manipulate their reports. We find that the CEO reports truthfully when her productivity type is too low to effectively mimic a type above \(\theta_1\). In particular, there is an endogenously determined threshold \(\theta_2\) such that CEO types below this threshold report truthfully. (The truthful reports from these low types entail certain replacement, which is unchanged from the model with two types.) This threshold \(\theta_2\), as well as \(\theta_1\), depend on the shareholders’ board policy \(c\). We provide conditions under which the shareholders prefer the board to be aggressive and then examine comparative statics. This analysis provides additional insights, such as predictions regarding the variation in CEO turnover under heightened performance uncertainty (variance of \(\varepsilon\)) and with greater uncertainty over the CEO’s type (variance of \(\theta\)).

C.1 Equilibrium with exogenous aggressiveness

As in the two-type setting, we begin the analysis of the continuous model where \(c\) is exogenously given and then examine the case of endogenously determined \(c\). In the second period, the manager has no incentive to misreport. This occurs for the same reason as in the two-type model; there is no replacement decision at that point and hence the manager cannot benefit from misreporting. The action selected in period 2 by a CEO of type \(\theta\) is thus \(a^2 = \omega^2(\theta)\). In the ensuing analysis, we consider incentives in the first period.

Also, we focus on the case of \(c \in [\bar{\theta} - E[\theta], \theta - E[\theta]]\). As we have seen in the two-type model (Theorem 1), only trivial equilibria emerge with extremely large or small \(c\). We state the result for the alternative case and then switch to the primary case of \(c \in [\bar{\theta} - E[\theta], \theta - E[\theta]]\).

**Proposition 2.** Suppose \(c < \bar{\theta} - E[\theta]\) or \(c > \theta - E[\theta]\). Then, the CEO always reports her true type. The board always retains the CEO if \(c < \bar{\theta} - E[\theta]\). When \(c > \theta - E[\theta]\), the board always replaces the CEO.

\(^{37}\)The two exemplary equilibria stated in Proposition 2 survive both D1 and D2 criteria because there is no out-of-equilibrium message.
Structure of equilibria

Like the two-type model, the CEO does not necessarily report her true type in the first period. Suppose that the board believes that reports are truthful. Then, by replacing the manager with report $\theta$, the board forgoes the period-2 expected output $\theta$ under the current manager but instead gains the subsidy or cost $c$ and the expected period-2 output $E[\theta]$ after replacement. Therefore, the board replaces the CEO when the report $\theta$ is less than $E[\theta] + c$, whereas the CEO with $\theta > E[\theta] + c$ remains in her position.

We naturally conjecture that the threshold $\theta_1 = E[\theta] + c$ determines the behavior of the manager in the following manner. The manager with type $\theta > \theta_1$ truthfully reports her type to the board, presumably because she does not need to hide her type in order to survive in the current firm. On the other hand, the CEO with type $\theta < \theta_1$ sometimes reports a message higher than $\theta_1$ in order to have a chance to remain in the firm.

Given that the reports above $\theta_1$ pool different types, the board sometimes must replace the CEO such that productive managers are retained with a higher likelihood than less productive ones. As in the model with two types, the optimal retention policy is a cutoff rule.

The derivation for the optimal cutoff is somewhat involved, so we refer readers to Appendix C.4 for the technical details and discussion regarding equilibrium selection. We find that the board employs a uniform cutoff $k_* \in [-\infty, +\infty]$ for all reports it receives from the CEO above $\theta_1$. In order to show this, we first prove that all reports above $\theta_1$ occur on the equilibrium path and that a CEO with type $\theta > \theta_1$ always reports truthfully. We state the result here and provide the details in Appendix C.4.

**Proposition 3.** In any equilibrium that survives the D1 criterion, the manager with type above $\theta_1$ truthfully reports her type, and the board (almost surely) employs an identical threshold level $k_*$ for replacement after observing any report above $\theta_1$.

Proposition 3 significantly simplifies the analysis in several ways. First, as mentioned above, all messages above $\theta_1$ are on the equilibrium path. Hence, we no longer need to worry about equilibrium selection on these high messages. Second, we can partition the type space $\Theta = (\underline{\theta}, \bar{\theta})$ into two disparate intervals: types above $\theta_1$ and types below. What remains is to analyze the behavior of CEO types in the interval $(\underline{\theta}, \theta_1]$.

Third, the manager with type $\theta \in (\underline{\theta}, \theta_1]$ will be replaced for sure unless she pretends to have a high type $\hat{\theta} \in (\theta_1, \bar{\theta})$. More precisely, it cannot occur (except on an event of probability 0) that the CEO has type $\theta \in (\underline{\theta}, \theta_1]$, reports $\hat{\theta} \in (\underline{\theta}, \theta_1]$, and is retained. Thus, we can essentially assume that types $\theta \in (\underline{\theta}, \theta_1]$ have only two choices: the truthful report $\theta$ or some misreport $\hat{\theta} \in (\theta_1, \bar{\theta})$. However, it is guaranteed only on the equilibrium path that the board replaces the manager for sure after message $\hat{\theta} \in (\underline{\theta}, \theta_1]$: the message $\hat{\theta} \in (\underline{\theta}, \theta_1]$ can be out of equilibrium, and after this message, the board may still form a modestly optimistic belief and set a cutoff that is not infinitely strict (i.e., $k(\hat{\theta}) < \infty$) but high enough to discourage every type from using this message. We ultimately show that such optimistic beliefs cannot survive equilibrium selection (Theorem 5). Meanwhile, we simply assume in the exposition that any report $\hat{\theta} \in (\underline{\theta}, \theta_1]$ results in the removal of the manager for sure (i.e., $k(\theta) = +\infty$).
In sum, we have characterized the structure of equilibria for types and reports higher than $\theta_1$. It is still unclear if, as expected, the board removes the CEO after receiving a message below $\theta_1$. We postpone the analysis of this question as it requires another stage of equilibrium selection. We instead investigate the reporting behavior of manager types below $\theta_1$, assuming that any report below $\theta_1$ certainly induces CEO replacement. We then return to the equilibrium selection problem (Theorem 5).

**Equilibrium decisions**

As in the model with two types, the manager with $\theta < \theta_1$ faces two choices. If the manager reports her true type, she obtains the informational gain $d$ but the board removes her before the next period with probability one. By reporting something above $\theta_1$, the manager forgoes the gain $d$ but has a positive chance to remain in the current firm. Since the probability of retention is $1 - F(k_{\ast} + d - \theta)$ in the latter case, the indifference condition is

$$\{1 - F(k_{\ast} + d - \theta)\} \chi = d.$$  

By solving this equation, we obtain the threshold type with which the manager is indifferent between the above two choices:

$$\theta_2(k_{\ast}) = k_{\ast} + d + F^{-1}(d/\chi).$$  \hspace{1cm} (C.1)

Types below this threshold $\theta_2$ cannot gain a satisfactory retention rate even after mis-reporting (i.e., $\{1 - F(k_{\ast} + d - \theta)\} \chi < d$). These types thus rather prefer to report truthfully. In contrast, types above $\theta_2$ but below $\theta_1$ prefer to inflate their report because $\{1 - F(k_{\ast} + d - \theta)\} \chi > d$. The following lemma summarizes this argument.\(^{38}\)

**Lemma 6.** Suppose $\theta < \theta_1$. In any equilibrium that survives the D1 criterion, the manager truthfully reports her type if $\theta < \theta_2$; and reports a message above $\theta_1$ if $\theta > \theta_2$.

The value of $\theta_2$ is thus the threshold such that types below this level report truthfully and are replaced with certainty. As we see shortly, shareholders can induce informative communication (i.e., truthful reports) from types lower than $\theta_2$ by raising this threshold $\theta_2$. This is achieved by setting a more aggressive board and consequently raising $\theta_1$ (at the expense of misreporting by intermediate types). This feature is analogous to the disciplinary effect we observed in the two-type model.

We then investigate how the uniform cutoff $k_{\ast}$ is determined by the board given this reporting behavior. We saw above that the cutoff levels $k(\hat{\theta})$ for reports $\hat{\theta} > \theta_1$ must be some uniform level $k_{\ast}$ (Proposition 3), but each cutoff level $k(\hat{\theta})$ needs to be a solution of the optimization problem for the board and thus depends on the posterior belief after observing report $\hat{\theta}$. Thus, if the posterior beliefs for such reports are not properly aligned—e.g., when certain messages attract too many (or too few) misreporting types—the board may employ several different cutoffs, which never occurs in equilibrium due to Proposition 3. In what

\(^{38}\)We allow $\theta_2 > \theta_1$ and $\theta_2 < \theta$. The latter case does not cause any problem as long as we set $g(\theta) = 0$ for $\theta < \theta$. We can easily see that $\theta_2 > \theta_1$ never occurs in equilibrium; if $k_{\ast}$ is so high that $\theta_2$ exceeds $\theta_1$, the manager never misreports her type and thus $k_{\ast}$ goes down to $-\infty$.  

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follows, we instead require the board to choose some uniform cutoff $k^*$ for (almost) every $\hat{\theta} > \theta_1$ and find a necessary condition for each value of $\hat{\theta}$. In this way, we can eventually obtain a single, useful condition that determines the value of $k_*$ as a function of $\theta_2$, and the function $k_*(\theta_2)$ works as if it is the board’s best response function.

For ease of exposition, we focus on an equilibrium where all types in the misreporting interval $(\theta_2, \theta_1)$ employ the same density function $h(\hat{\theta})$ in choosing a misreport $\hat{\theta}$.

We first calculate the posterior belief of the board after observing $\hat{\theta} \in (\theta_1, \theta_2)$:

$$\text{Prob}\{\theta = \hat{\theta} \mid \hat{\theta}\} = \frac{g(\hat{\theta})}{g(\hat{\theta}) + h(\hat{\theta}) \int_{\theta_2}^{\theta_1} g(\theta) \, d\theta} = \frac{g(\hat{\theta})}{Q(\hat{\theta})}$$

and,

$$\text{Prob}\{\theta \neq \hat{\theta} \mid \hat{\theta}\} = \frac{h(\hat{\theta}) \int_{\theta_2}^{\theta_1} g(\theta) \, d\theta}{g(\hat{\theta}) + h(\hat{\theta}) \int_{\theta_2}^{\theta_1} g(\theta) \, d\theta} = \frac{h(\hat{\theta}) \int_{\theta_2}^{\theta_1} g(\theta) \, d\theta}{Q(\hat{\theta})},$$

where $Q(\hat{\theta}) = g(\hat{\theta}) + h(\hat{\theta}) \int_{\theta_2}^{\theta_1} g(\theta) \, d\theta$ represents the probability (density) that the manager chooses report $\hat{\theta}$. Also note $\int_{\theta_2}^{\theta_1} g(\theta) \, d\theta$ is the unconditional probability of misreporting.

More specifically, the posterior probability of $\theta \leq x$ is

$$\text{Prob}\{\theta \leq x \mid \hat{\theta}\} = \frac{h(\hat{\theta}) \int_{\theta_2}^{\theta_1} g(\theta) \, d\theta}{Q(\hat{\theta})},$$

for all $x \in (\theta_2, \theta_1)$. Hence, type $\theta \in (\theta_2, \theta_1)$ has a density $h(\hat{\theta})g(\theta)/Q(\hat{\theta})$ conditional on report $\hat{\theta}$, while the truthful type $\theta = \hat{\theta}$ has probability $g(\hat{\theta})/Q(\hat{\theta})$ as an atom.

Given the above posterior belief, the board must be indifferent between keeping and replacing the manager after observing output $y = k_*$. The corresponding indifference condition is

$$\hat{\theta} \cdot f(k_* - \hat{\theta}) \frac{g(\hat{\theta})}{Q(\hat{\theta})} + \int_{\theta_2}^{\theta_1} \theta \cdot f(k_* + d - \theta) \frac{h(\hat{\theta})g(\theta)}{Q(\hat{\theta})} \, d\theta = \theta_1,$$

conditional prob. of $\hat{\theta}$

conditional density of $\theta$

or equivalently,

$$(\hat{\theta} - \theta_1) f(k_* - \hat{\theta}) g(\hat{\theta}) = h(\hat{\theta}) \int_{\theta_2}^{\theta_1} (\theta_1 - \theta) f(k_* + d - \theta) g(\theta) \, d\theta. \quad (C.2)$$

This condition guarantees the optimality of cutoff $k$ for each individual report $\hat{\theta}$.

By integrating this individual-level condition (C.2) with respect to $\hat{\theta}$, we obtain an ag-

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\text{Such an equilibrium always exists, but many other equilibria also exist. See the proof of Lemma 7 for the general case.}
necessary condition for optimality:
\[
\int_{\theta_1}^{\theta} (\theta - \theta_1)f(k_* - \theta)g(\theta) \, d\theta = \int_{\theta_2}^{\theta_1} (\theta_1 - \theta)f(k_* + d - \theta)g(\theta) \, d\theta.
\]
(C.3)

The left-hand side represents the aggregate positive effect from keeping the current manager across all types above \(\theta_1\). The right-hand side is the corresponding effect from the misreporting types. When \(\theta_2\) is given, the equilibrium value of the uniform cutoff \(k_*\) must satisfy the necessary condition (C.3). Indeed, we uniquely find the value of \(k_*\) that solves (C.3) due to the monotone likelihood property. The following lemma summarizes the above argument and provides additional results.

**Lemma 7.** In any equilibrium that survives the D1 criterion, the uniform cutoff \(k_*\) for reports above \(\theta_1\) satisfies condition (C.3). For each value of \(\theta_2 \in [-\infty, \theta_1)\), there uniquely exists \(k_*(\theta_2) \in \mathbb{R}\) that solves condition (C.3). The function \(k_*(\theta_2)\) is continuous, non-increasing, and goes to \(-\infty\) as \(\theta_2 \to \theta_1\).

To construct an equilibrium, we aim to find a pair \((\theta_2^f, k^f)\) that simultaneously satisfies \(\theta_2^f = \theta_2(k^f)\) and \(k^f = k_*(\theta_2^f)\). To this end, we consider a function
\[
\Gamma(k) = \begin{cases} 
 k_*(\theta_2(k)) & \text{if } \theta_2(k) < \theta_1 \\
 -\infty & \text{otherwise},
\end{cases}
\]
and its fixed point. The function \(\Gamma\) is non-increasing because \(k_*\) is non-increasing and \(\theta_2\) is increasing. The maximum \(\Gamma(-\infty) = k_*(\theta)\) is finite and the minimum \(\Gamma(+\infty) = k_*(\theta_1)\) goes to \(-\infty\). Since \(\Gamma\) is continuous, we can find a unique fixed point \(k^f\). The fixed point \(k^f\) is finite and thus \(\theta_2^f\) is smaller than \(\theta_1\).

**Proposition 4.** The function \(\Gamma(k)\) has a unique, finite fixed point \(k^f\). Define \(\theta_2^f = \theta_2(k^f)\). Then, \(\theta_2^f < \theta_1\) and the pair \((\theta_2^f, k^f)\) satisfies \(k^f = k_*(\theta_2^f)\) as well as \(\theta_2^f = \theta_2(k^f)\).

Due to Proposition 4, any equilibrium that survives the D1 criterion must use \(k^f\) as the uniform cutoff and \(\theta_2^f\) as the threshold \(\theta_2\) of misreporting. Conversely, we can also construct an equilibrium from these two parameters. (Equation (C.2) constructs an equilibrium by determining the equilibrium value of \(h(\hat{\theta})\).) We have fully characterized (except on the set of probability 0) the behavior of the board and the manager on the equilibrium path, but it remains unknown whether the board sets \(k(\hat{\theta}) = +\infty\) after out-of-equilibrium messages between \(\theta_2^f\) and \(\theta_1\). Although this out-of-equilibrium behavior is now irrelevant in characterizing what happens on the equilibrium path, we can obtain the desired result—an infinite cutoff for bad messages—by imposing the D2 criterion.\(^{40}\)

**Theorem 5.** Consider the model with exogenous \(c \in \left[\bar{\theta} - \mathbb{E}(\theta)\right]\). There exists a perfect Bayesian equilibrium that survives both the D1 and D2 criteria. Any equilibrium that survives the D1 criterion almost surely satisfies the following properties in the first period:

\(^{40}\)The D2 criterion imposes more restrictions than the D1 criterion. Thus, if an equilibrium survives the D2 criterion, then this equilibrium also survives the D1 criterion. See fn. 49 regarding why the D1 criterion is insufficient.
The manager truthfully reports her type if $\theta > \theta_1$ or $\theta < \theta_2^f$. The manager with type $\theta \in (\theta_2^f, \theta_1)$ chooses some report above $\theta_1$.

The board replaces the manager when the manager reports $\hat{\theta} < \theta_1$ or the cash flow $y$ is less than $k^f$. The board retains the manager if $\hat{\theta} > \theta_1$ and $k > k^f$.

Here, $\theta_2^f$ and $k^f$ are as in Proposition 4. Furthermore, in any equilibrium that survives the D2 criterion, the board sets cutoff $+\infty$ after almost every report $\hat{\theta} \in (\theta_2^f, \theta_1)$.

We make three remarks before proceeding to endogenize the parameter $c$. First, the CEO’s reporting behavior is non-monotonic as depicted in Figure 5. This non-monotonicity occurs because the CEO with low $\theta$ is unable to get a reasonably high chance of retention when she mimics some type. Hence, types below $\theta_2$ truthfully reveal their ability, learn the true state, and then are subsequently replaced. The threshold type $\theta_2$’s retention probability from mimicking is high enough that she is indifferent between truthful reporting and mis-reporting. The types above $\theta_2$ but below $\theta_1$ overstate their types, sometimes far above $\theta_1$. The types above $\theta_1$ report truthfully.

Second, the board employs a two-step retention policy. As the first step, the manager asks the CEO to report her current situation. If the CEO reports something too pessimistic, the board helps her to revive the firm during her tenure but the removal of the CEO is unchangeable. This corresponds to the findings of Cornelli et al. (2013), who show that boards often utilize “soft” (nonverifiable) information regarding the CEO’s ability when making replacement decisions. Having passed the first step, to stay in the firm, the CEO needs to achieve the target $k_*$ set by the board, as the second step. This equilibrium replacement behavior helps to explain the inverse relationship between performance and CEO turnover found in the empirical literature (see, e.g., Jenter and Lewellen (2017)).

Third, managers with intermediate ability $\theta \in (E[\theta], \theta_1)$ are sometimes removed due to poor communication with the board (i.e., misreporting). Because truthful reporting terminates her tenure, these CEOs are urged to overstate their situation. This miscommunication reduces the effectiveness of the advice from the board and, consequently, CEOs with ability in this range tend to have worse performance due to the lack of information.

C.2 Shareholders’ decision

Due to the complexity of the model with a continuous type space, we must employ additional distributional assumptions in order to obtain analytic results with endogenously determined $c$. Specifically, we assume that type $\theta$ and noise $\varepsilon$ are uniformly distributed on supports $[\bar{\theta}, \tilde{\theta}]$ and $[-q, q]$, respectively. Although the uniform distribution $F(\varepsilon)$ does not fully satisfy the monotone likelihood ratio property, the distribution can be seen as a limit of distributions with this property.\footnote{Here is an example of such a sequence. Let $\phi$ denote the density of the standard normal distribution and define $z_n(x)$ by $z_n = |x|/n$ for $|x| < q$ and by $z_n = n|x| - q(n - 1/n)$ for other $x$. Then, density $f_n(x) = \phi(z_n(x))/\int_{-\infty}^{\infty} \phi(z_n(y))dy$ satisfies the monotone likelihood ratio property, because $z_n$ is convex, and the corresponding distribution $F_n(x) = \int_{-\infty}^{x} f_n(x)dx$ weakly converges to the uniform distribution.}

We note that the results are not qualitatively sensitive to these assumptions, as shown in the simulations reported in Appendix C.3.

One benefit of using the uniform distribution is that we can calculate closed-form characterizations of the board’s cutoff strategy and the shareholders’ optimal board policy. One
drawback, however, of the uniform setting is that certain pathological cases arise which confound the analysis. We thus impose the following regularity conditions:

\[
\frac{2q}{\chi} > \frac{4q - \Delta}{2\Delta}, \tag{C.4}
\]

where \(\Delta = \bar{\theta} - \theta\). This condition ensures that the distortion effect in the cutoff (i.e., too strict cutoff) is not too strong. We later see that \(2q/\chi\) has a negative effect on \(k - \theta_2\). When \(\theta_2\) is fully determined by other parameters—this is the case in Proposition 5 (i)—an increase in \(2q/\chi\) lowers the level of equilibrium cutoff and thus mitigates the distortion in the retention decision. We also impose the following condition:

\[
\theta + q - (1 + 8q/\chi)^2 d \geq \bar{\theta} - q. \tag{C.5}
\]

This inequality is a technical condition that significantly simplifies the analysis by eliminating subtle pathological cases that arise in the dual uniform setting. To interpret this condition, note that the inequality (C.5) implies \(\theta + q \geq \bar{\theta} - q\); that is, the support of \(\varepsilon\) is sufficiently large such that low-type CEOs can potentially mimic up to the highest type.

We first characterize the equilibria with exogenously given \(c\) for this parameterization. We focus on equilibria consistent with the equilibria found in Appendix C.1: The board employs a uniform cutoff \(k\) for messages above \(\theta_1\) and replaces the manager for sure with messages below \(\theta_1\). To simplify the notation, let \(\theta_{2,0} = d + F^{-1}(d/\chi) \equiv d - q + 2dq/\chi\) denote the intercept of the threshold \(\theta_2 = k + \theta_{2,0}\) (see equation (C.1)).

**Proposition 5.** Consider the game described above with exogenously given \(c\), and we focus on the class of equilibria described above. Assume the regularity conditions (C.4) and (C.5).

(i) Suppose \(c \in (0, \Delta/2)\). In any equilibrium, the board sets a uniform cutoff \(k = 2\theta_1 - \bar{\theta} - \theta_{2,0}\) and the threshold \(\theta_2 = 2\theta_1 - \bar{\theta}\) is greater than the worst type \(\bar{\theta}\). The cutoff is high enough to replace even the best type with positive probability (i.e., \(\bar{\theta} - q \leq k\)).

(ii) Suppose \(c \in (-\Delta/2, 0)\). In any equilibrium, the board sets a uniform cutoff \(k = 2\theta_1 - q - \bar{\theta}\) and no type below \(\theta_1\) chooses a truthful message (i.e., \(\theta_2 \leq \bar{\theta}\)). The cutoff is low enough such that the best type is never replaced (i.e., \(\bar{\theta} - q \geq k\)).

(iii) Suppose \(c = 0\). In any equilibrium, no type below \(\theta_1\) chooses a truthful message (i.e., \(\theta_2 \leq \bar{\theta}\)). The neutral board has continuously many optimal cutoffs and the set of optimal cutoffs is the interval between the cutoffs given in (i) and (ii); i.e., \([\bar{\theta} - q, \bar{\theta} - \theta_{2,0}]\).

(iv) Suppose \(|c| \geq \Delta/2\). In any equilibrium, the manager truthfully reports her type for sure. The board replaces the manager with probability 1 if \(c \geq \Delta/2\). The manager is retained for sure if \(c \leq -\Delta/2\).

For any value of \(c\), an equilibrium exists.

The first two cases are especially important. In case (i), types above \(\theta_1\) are rare so that the board needs to bring the cutoff \(k\) high enough to discourage lower types from mimicking. As a result, the worst types report truthfully but even the best type faces the risk of replacement. In equilibrium, the misreporting interval between \(\theta_1\) and \(\theta_2\) needs to be perfectly balanced with the types above \(\theta_1\)—due to the uniform specification—so that \(\bar{\theta} - \theta_1 = \theta_1 - \theta_2\). This value of \(\theta_2\), in turn, determines the value of \(k = \theta_2 - \theta_{2,0}\).
In contrast, the board in case (ii) faces a large mass of types above $\theta_1$ so that the cutoff is too low to discourage imitation. Consequently, some types above $\theta_1$ are never replaced with the friendly choice of a cutoff. The threshold is type $\theta = k + q$; types above the threshold always have output higher than the cutoff. The equilibrium condition in this case is thus $(k + q) - \theta_1 = \theta_1 - \bar{\theta}$.

Cases (iii) and (iv) are less important in two different senses. Case (iv) is a trivial case where the board employs an extremely aggressive or friendly retention policy. The equilibrium multiplicity in case (iii) is apparently problematic, but the choice of $k$ does not affect the payoff for the shareholders because the neutral board perfectly represents the shareholders’ interest. Therefore, we only need to analyze the first two cases, keeping in mind that the extreme board (case (iv)) could be optimal. Indeed, we ultimately show that the optimal choice of $c$ always lies in case (i).

We now aim to find the optimal level of $c$ for shareholders. As in the two-type model, the shareholders’ payoff can be divided into the two periods. In the first period, shareholders receive the payoff

$$V_1 = -d \left\{ G(\theta_1) - G(\theta_2) \right\},$$

plus $E[\theta]$. Here, the difference $G(\theta_1) - G(\theta_2) = \Pr\{\theta \in [\theta_2, \theta_1]\}$ is the probability of misreporting. The CEO in period 2 never misreports her type, but the board’s retention policy in the first period affects the expected value of $\theta$ in the second period. The expected second-period value is

$$V_2 = \int_{\theta_1}^{\bar{\theta}} (\theta - E[\theta]) F(\theta - k) g(\theta) \, d\theta + \int_{\theta_2}^{\theta_1} (\theta - E[\theta]) F(\theta - k - d) g(\theta) \, d\theta,$$

(C.6)

plus $E[\theta]$. The shareholders maximize the sum $V = V_1 + V_2$ by controlling $c$.

We can analytically calculate the values of $V_1$ and $V_2$ in this dual uniform environment due to Proposition 5. In the exposition, we focus on the relevant case of $c \in (0, \Delta/2)$. (See the Appendix for the case of the friendly board.) Since $\theta_2 = 2\theta_1 - \bar{\theta} = E[\theta] + 2c - \Delta/2$, the period-1 payoff $V_1$ is

$$V_1 = -d \left( \frac{1}{2} - \frac{c}{\Delta} \right).$$

(C.7)

The period-2 payoff $V_2$ is similarly given as:

$$V_2 = \int_{\theta_1}^{\bar{\theta}} (\theta - E[\theta]) \left( \frac{\theta - k + q}{2q} \right) \frac{d\theta}{\Delta} + \int_{\theta_2}^{\theta_1} (\theta - E[\theta]) \left( \frac{\theta - d - k + q}{2q} \right) \frac{d\theta}{\Delta}.$$

By $k = 2\theta_1 - \bar{\theta} - \theta_{2,0} = E[\theta] + 2c - \Delta/2 - \theta_{2,0}$, we obtain the following cubic function:

$$V_2 = \frac{1}{48q} \left( 1 - \frac{2c}{\Delta} \right) \left\{ 24c \cdot \theta_{2,0} + 4c(\Delta + 6q) + 2\Delta^2 - d(18c - 3\Delta) - 16c^2 \right\}.$$

(C.9)

Note that $1 - 2c/\Delta > 0$ because $c < \Delta/2$. 

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Since $V = V_1 + V_2$ is a cubic function with a positive coefficient on $c^3$, the function $V$ will increase, attain a local maximum, decrease, and then increase again if $V$ behaves regularly enough. Such non-monotonicity occurs due to the countervailing disciplinary and distortion effects. Recall that these two effects first emerged in the analysis of the two-type setting in Section 3. The disciplinary effect from raising $c$ results in low-type CEOs reporting truthfully to the board. In this continuous setting, the distortion effect not only appears in the form of an excessively strict cutoff, but also emerges as an inefficiently high standard in reporting. That is, the board demands highly optimistic reports for retention considerations, and consequently, intermediate types $\theta \in (E[\theta], \theta_1)$ must leave their position after truthful reporting.

After a few calculations, we obtain that the local maximum is attained at

$$c^*_A = \frac{1}{8} \left\{ d \left( 1 + \frac{8q}{\chi} \right) + 2\Delta - \sqrt{D} \right\}, \quad \text{(C.10)}$$

where $D = d^2 (1 + 8q/\chi)^2 + 4d\Delta - 32dq + 4\Delta^2 > 0$. In the Appendix, we show that this local optimum is indeed the global optimum.

**Theorem 6.** Assume the regularity conditions (C.4) and (C.5). It is optimal for shareholders to choose an aggressive board with $c \in (0, \Delta/2)$. The optimal value of $c$ is uniquely given by equation (C.10).

Theorem 6 states that shareholders set the board to be aggressive, and provides a closed-form characterization of the optimal policy $c$. We note that, in contrast to the two-type case, the optimal policy is not maximal aggression (i.e., $c \in (0, \Delta/2)$). We find that a moderately aggressive board is optimal since the shareholders’ disutility from the distorted retention decision eventually exceeds the benefit of the disciplinary effect as the degree of aggressiveness, $c$, increases.

This result is in stark contrast to several theoretical studies which have found that a management-friendly board (excessive retention) is optimal for shareholders. By considering the interrelationship between advising and replacement, we find that aggressive boards (excessive replacement) can be optimal for shareholders, which upends the results of models that separately examine advising (e.g., Adams and Ferreira (2007)) or replacement (e.g., Almazan and Suarez (2003)).

Furthermore, the explicit solution $c^*_A$ allows for comparative statics analysis. Note that the variables $q$ and $\Delta$ are interchangeable with the variances of noise $\varepsilon$ and type $\theta$, respectively, in the comparative statics below, because $\text{Var}(\varepsilon) = q^2/3$ and $\text{Var}(\theta) = \Delta^2/12$.

**Proposition 6.** Assume the regularity conditions (C.4) and (C.5). The optimal aggressiveness $c^*_A$ is increasing in the cost of miscommunication, $d$, and in the variance of noise $\varepsilon$, and decreasing in the variance of type $\theta$. An increase in the private benefits, $\chi$, decreases (increases) $c^*_A$ if $\Delta - d(8q - \Delta)$ is positive (negative).

An increase in $d$, the value loss from uninformative communication, results in shareholders setting a more aggressive board. Intuitively, this occurs since shareholders prefer to elicit greater truthful reporting, and thus informative communication, from the CEO. Indeed, the
coefficient on $c$ in $V_1$ increases (see equation (C.7)). On the other hand, an increase in $d$ also decreases the cutoff $k$, as seen in Proposition 5 (i), by making truthful communication more attractive and misreporting less likely. Consequently, to compensate for this milder replacement threshold by the board and due to their strengthened preference for truthful reporting, shareholders push to make the board more aggressive in their replacement of the CEO as $d$ increases.

Similarly, shareholders prefer a more aggressive board as the variance of noise (and thus $q$) increases. In this case, the support of $\varepsilon$ expands and more low-type managers are potentially able to mimic a higher type in their observed output. Likewise, there is less room for very high types to meet the cutoff with certainty. As a result, the cutoff-based retention policy becomes less effective and the choice of cutoff $k$ becomes less important (indeed, $V_2$ shrinks as $q$ increases; see (C.9)). In other words, the increase in the noise dilutes the distortion effect of aggressiveness, while the disciplinary effect, represented by $V_1$, is unchanged. With the disciplinary effect intact but the distortion effect weakened, shareholders face an incentive to make the board more aggressive in response to an increase in $q$.

Proposition 5 also shows that the shareholder’s optimal aggressiveness $c_A^*$ is decreasing in the variance of the CEO’s productivity $\theta$ (and thus $\Delta$). As the variance increases, the shareholders face an increased risk in replacing a highly talented CEO. We find that the increase in variance amplifies the disutility from the distortion effect of a high cutoff $k$, and leads shareholders to prefer a comparatively less aggressive board.

Lastly, the effect of an increase in the private benefits $\chi$ is negative when $\Delta$ (or equivalently, the variance of productivity $\theta$) is sufficiently large compared to $d$ and $q$ (or equivalently, the variance of noise $\varepsilon$). As the private benefit $\chi$ increases, misreporting becomes more appealing for a low-type manager. The board, in turn, responds to the CEO’s increased incentive for mimicry by increasing the retention standard $k$. The shareholders do not favor this decision of raising the bar when $\Delta$ is large enough. As already seen in the previous paragraph, an increase in $\Delta$ worsens the distortion effect of an aggressive board. Hence, the board’s response to increasing the cutoff $k$ is an overreaction from the shareholders’ perspective, thus resulting in a decrease in $c_A^*$. Conversely, the distortion effect becomes relatively unimportant in a highly noisy situation (when $q$ is high relative to $\Delta$), which is also already seen two paragraphs above. This induces the shareholders to make the board more aggressive in response to the increased misreporting incentive from a higher $\chi$.

C.3 Numerical Results

In this appendix, we present numerical exercises for additional economic implications and to show that the results of the model are robust to alternative distributions of $\theta$ and $\varepsilon$. We first assume that the noise $\varepsilon$ is normally distributed and type $\theta$ is exponentially distributed. More specifically, the distribution of $\varepsilon$ has mean 0 and variance 1 and the exponential distribution has intensity $\lambda = 1$. Also, we use $d = 1$ and $\chi = 2$ throughout the numerical analyses.

As shown in Figure 6, the optimal board is moderately aggressive (note that $\theta_1 = 1 + c$); the optimal value of $c$ is numerically given as $c \approx 0.4775$. The bottom of the negative spike in Figure 6 represents the point where $\theta_2$ reaches $\overline{\theta}$ ($= 0$). This sharp drop in the shareholders’ payoff occurs due to frequent misreporting triggered by the board’s excessively friendly retention policy. Figure 7 elucidates this point by describing how misreporting
behavior changes in response to a change in \( \theta_1 = \mathbb{E}[\theta] + c \). The bottom of the negative spike in Figure 6 corresponds to the peak of the spike in the top panel of Figure 7 (i.e., when the misreporting probability is greatest). The misreporting region and the interval initially increase as \( \theta_1 \) increases, and is maximized when \( \theta_2 \) reaches \( \frac{\theta}{2} \) (i.e., \( \theta = 0 \)). After this point, the misreporting interval \([\theta_2, \theta_1]\) is pushed leftward and exponentially reduces its probability while keeping the width \( \theta_2 - \theta_1 \) constant. This disciplinary effect—the reduction of misreports due to the board’s aggressive retention policy—creates the hump after the negative spike in Figure 6 and induces a moderately aggressive board to be optimal for shareholders.

To see how the other effect—distortion in retention decisions—hurts the value of the firm for shareholders, see Figure 8 which depicts the contribution of the equilibrium retention policy to the shareholders’ objective function. To gauge the level of distortion in the equilibrium retention policy, we introduce the optimal cutoff \( k_{\text{opt}}(\theta_1, \theta_2) \) with the misreporting interval \([\theta_2, \theta_1]\) as given. Formally, the optimal cutoff \( k_{\text{opt}}(\theta_1, \theta_2) \) is the solution of the maximization problem

\[
\int_{\theta_1}^{\theta_2} (\theta - \mathbb{E}[\theta]) F(\theta - k) g(\theta) d\theta + \int_{\theta_2}^{\theta_1} (\theta - \mathbb{E}[\theta]) F(\theta - k - d) g(\theta) d\theta.
\]

The corresponding first-order condition

\[
\int_{\theta_1}^{\theta_2} (\theta - \mathbb{E}[\theta]) f(\theta - k_{\text{opt}}) g(\theta) d\theta = \int_{\theta_2}^{\theta_1} (\mathbb{E}[\theta] - \theta) f(\theta - k_{\text{opt}} - d) g(\theta) d\theta.
\]

Interestingly, Figure 7 shows that, while the probability of misreporting declines as \( \theta_1 \) increases, the misreporting interval \([\theta_2, \theta_1]\) remains constant after the maximum is reached. This occurs because the board is shifting the misreporting interval \([\theta_2, \theta_1]\) further away from the high-density regions populated with low types.
Figure 7: Two variables that capture the misreporting behavior of the manager: the probability of misreporting (i.e., $G(\theta_1) - G(\theta_2)$) and the width of the misreporting interval $[\max\{\theta_2, \bar{\theta}\}, \theta_1]$.

resembles but differs from the equilibrium condition (C.3), in which the board uses $\theta_1 = \mathbb{E}[\theta] + c$ in place of $\mathbb{E}[\theta]$. As shown in the lower half of Figure 8, the equilibrium cutoff is suboptimal unless $c = 0$: the board is too strict in setting $k$ when aggressive (i.e., $c > 0$) and too lenient when friendly (i.e., $c < 0$).

Although Figure 8 shows that the equilibrium cutoff significantly deviates from the optimal level, this figure also suggests that the effect to the value of the firm is limited on the aggressive side (i.e., $c > 0$). To see why this is the case, we examine how the board’s aggressive cutoff impacts the second-period value $V_2$ of the firm, defined in equation (C.6). We decompose this effect by considering three groups: the truth-telling top group $(\theta_1, \theta_2)$, misreporting intermediate group $(\mathbb{E}[\theta], \theta_1)$, and misreporting bottom group $(\theta_2, \mathbb{E}[\theta])$. (The types below $\theta_2$ are the real bottom group, but they are replaced for sure anyway and thus not affected by the uniform cutoff.)

According to Figure 9, the suboptimality of the equilibrium cutoff slightly affects the bottom group $(\theta_2, \mathbb{E}[\theta])$ but has virtually no effect on the top and middle groups when $\theta_1 > \mathbb{E}[\theta]$ is close enough to $\mathbb{E}[\theta]$. Indeed, the effect on the bottom group is positive because a high cutoff helps to remove unwanted types in this group. Instead, the suboptimally high cutoff decreases the period-2 values from the top two groups, but the top group appears almost unaffected. The effect to the middle group is also minute (although it appears to be large due to the scaling of the graph). This observation does not change even if we replace the optimal cutoff with the first-best, but infeasible, retention policy: $k = +\infty$ for the bottom group and $k = -\infty$ for the top two groups. This observation implies that the
Figure 8: The gain from the manager selection ($V_2 - \mathbb{E}[\theta]$) and the choice of cutoff in the equilibrium retention policy. The solid curves represent the actual values in equilibrium. The dashed curves correspond to the optimal cutoff $k^{opt}(\theta_1, \theta_2)$ with $\theta_1$ and $\theta_2$ fixed.
Figure 9: The contribution of groups \((\theta_1, \overline{\theta}), (E[\theta], \theta_1), (\theta_2, E[\theta])\) to the normalized period-2 value of the firm. The solid curves represent the actual equilibrium values. The dashed curves represent the values with the optimal cutoff \(k^{\text{opt}}(\theta_1, \theta_2)\). The dotted curves represent those with the infeasible, first-best retention policy (i.e., \(k = -\infty\) for the top two groups and \(k = +\infty\) for the bottom group).
types in the top group can easily pass the equilibrium cutoff, which is inflated upwards due to $c > 0$. Also, the contribution from the middle group is negligibly small even with the most favorable cutoff $k = -\infty$. This negligibility is partly because the upward bias of the equilibrium cutoff increases $\theta_2$ and significantly lowers the probability of misreporting, as we have also seen in Figure 7.

We obtain similar results when $\theta$ is normally distributed. Figure 10 indicates that the optimal board is, once again, aggressive. In this figure, we assume that $\theta$ is normally distributed with mean 10 and standard deviation 1. All of the other parameters are the same. Although this case is more smooth than the exponential case—there is no longer a negative spike in the payoff or a discontinuity in the type distribution—we still observe similar patterns in Figure 11. The probability of misreporting is decreasing in $\theta_1$ in the aggressive region $\theta_1 > E[\theta]$ and the distortive impact of aggressiveness in the cutoff is quite limited.

These two numerical results exemplify the robustness of the results. Here, we present only two numerical results, but an aggressive board easily turns out to be optimal as long as the parameters are not too extreme. Even though analytic calculations are intractable, except for the dual uniform environment we have studied in Appendix C.2, the numerical results presented in this appendix prove how commonly aggressive boards emerge in our setting.

### C.4 Additional details on continuous types

In this appendix, we present additional details regarding the analysis of the setting with continuous types. In particular, we discuss the determination of the uniform cutoff rule under exogenously specified $c$. The first result establishes that any retention policy must be a cutoff strategy.

**Lemma 8.** Let $\hat{\theta} \in (\theta_1, \theta)$. After observing report $\hat{\theta}$, the board’s optimal retention policy is a cutoff rule with threshold $k(\hat{\theta}) \in [-\infty, +\infty]$ regardless of the posterior belief $\beta_{\hat{\theta}}$.

Observe that the cutoff $k(\hat{\theta})$ needs to be uniform across messages that the CEO uses when misreporting her type. When reporting $\hat{\theta} \neq \theta$, the manager with type $\theta$ only cares about the survival probability

$$
\Pr\{\theta - d + \epsilon > k(\hat{\theta})\} = 1 - F(k(\hat{\theta}) + d - \theta),
$$

which is decreasing in $k(\hat{\theta})$. Thus, the CEO always chooses a message $\hat{\theta}$ with the lowest cutoff $k(\theta) = \inf_s k(s)$ and never uses $\hat{\theta}$ with a higher cutoff when misreporting.

We claim that all reports above $\theta_1$ have the same cutoff level. It is still potentially possible at this stage that some or even all reports above $\theta_1$ are out-of-equilibrium and, after such reports, the board has a very pessimistic belief and an extremely strict cutoff. We temporarily allow such implausible beliefs and cutoffs in the next lemma (Lemma 9). However, we soon claim a fuller statement (Lemma 11) after equilibrium selection (Lemma 10).

**Lemma 9.** In any equilibrium with exogenous $c$, it occurs with probability 1 that, whenever the manager with type $\theta$ chooses a misreport $\hat{\theta} \in (\theta_1, \theta) \setminus \{\theta\}$, the cutoff $k(\hat{\theta})$ associated with the misreport is equal to $\inf_{s \in (\theta_1, \theta)} k(s)$.
Figure 10: The objective function when $\theta \sim N(10, 1)$.

Figure 11: The counterparts of Figures 7 and 8 when $\theta \sim N(10, 1)$.
As in the case of two types, we employ the D1 criterion (see Appendix A) to eliminate the anomaly associated with out-of-equilibrium reports and to ensure truthful reporting from types $\theta > \theta_1$. As we see soon, when the CEO reports an out-of-equilibrium message $\hat{\theta} \in (\theta_1, \infty)$, the board must believe that the CEO’s type is above $\theta_1$ after we apply the D1 criterion. If this is the case, the board sets $k(\hat{\theta}) = -\infty$ (i.e., no replacement) and consequently the type $\hat{\theta}$ (as well as many other types) begins to use the message $\hat{\theta}$ to utilize the extremely friendly retention policy; consequently, message $\hat{\theta}$ is no longer out-of-equilibrium. Once out-of-equilibrium messages disappear from the interval $(\theta_1, \theta)$, all CEO types in this interval report truthfully and face the uniform cutoff $k^* = \inf_s k(s)$. (See the proof of Lemma 10 for details.)

To illustrate the D1 criterion in the present setting, consider an (ideal) equilibrium where the manager always encounters a uniform cutoff $k^*$ after misreporting her type on the equilibrium path.\footnote{This simple structure may not arise in a presumably implausible equilibrium where some types above $\theta_1$ choose messages lower than $\theta_1$. We eliminate such pathological cases in the proof of Lemma 10.} We aim to show that if $\hat{\theta} \in (\theta_1, \theta)$ is an out-of-equilibrium message, then it is type $\hat{\theta}$ that benefits the most from this message among all other types. First, observe that the manager with type $\theta$ can get at least the following payoff from misreporting:

$$U^*(\theta) = (1 - F(k^* + d - \theta))\chi + \{2\theta + \chi\},$$

in equilibrium under cutoff $k_*$. To guarantee this payoff or better, the out-of-equilibrium message $\hat{\theta}$ must result in a cutoff $k(\hat{\theta}) \leq k_*$ if the manager has type $\theta \neq \hat{\theta}$. When $\theta = \hat{\theta}$, the type $\hat{\theta}$ receives a much higher payoff with the truthful report $\hat{\theta}$ than the equilibrium payoff $U^*(\hat{\theta})$, as this type can uniquely boost her output $y$ with the message $\hat{\theta}$. As a result, by the D1 criterion, the type $\hat{\theta}$ is the only type that deserves a probability weight. The actual proof is somewhat more involved than the above discussion.\footnote{In the derivation, we cannot assume that all misreporting types face some cutoff rule; Lemma 8 applies only to messages above $\theta_1$, and the other messages may induce intractable retention policies. Nevertheless, thanks to Lemma 8, at least the retention policy after $\hat{\theta}$ is tractable even though the other side—the retention policy each type faces in equilibrium—may be pathological.} We ultimately obtain the following result:

**Lemma 10.** In any equilibrium that survives the D1 criterion, the manager with type above $\theta_1$ truthfully reports her type.

We now know that no report above $\theta_1$ is an out-of-equilibrium message. In other words, we have overcome the problem of implausibly pessimistic beliefs and can strengthen the statement of Lemma 9.\footnote{We actually prove Lemma 11 in the proof of Lemma 10.}

**Lemma 11.** In any equilibrium that survives the D1 criterion, the board (almost surely) employs cutoff $k_* = \min_{\hat{\theta} \in (\theta_1, \theta)} k(\hat{\theta})$ after receiving a report above $\theta_1$. 

\[\]
D Continuous retention decision

In our baseline model, the board makes a retention decision, which is naturally binary. In this appendix, we investigate how this binary specification of the board’s decision contributes to this paper’s results, by considering a setting where the board makes a continuous retention decision. More specifically, we consider a setting where the board makes a decision concerning the retention of the CEO but the retention decision is not binary. To make possible this paradoxical statement—whether to retain or replace the CEO is naturally binary—we assume that the board chooses \( z \in [0, 1] \) as the probability of replacement, instead of directly choosing whether to retain or replace the CEO. If the payoff gain (or loss) from replacement is unchanged from \( z \cdot c \), this change makes no difference from the baseline model because the newly added choices \( z \in (0, 1) \) never become optimal. We hence allow the board to have a nonlinear gain (or loss) \( B(z, c) \) from replacing the CEO.

In order to prevent corner solutions, we consider the case where \( B(z, c) \) diverges to \(-\infty\) as \( z \) goes to zero or one. Specifically, we consider the structure where \( B(z, c) = z \cdot c + h(z) \) such that \( h(0) = h(1) = -\infty \). To simplify the argument, \( h(z) \) is a single-peaked function with \( h'(0) = \infty, h'(1) = -\infty, \) and \( h''(z) < 0 \) for all \( z \in (0, 1) \). For example, \( h(z) = \log(z - z^2) \) satisfies all of these conditions.

We derive the optimal choice of \( z \). When the board believes the probability of the high type is \( \mu \), the decision problem of the board is to maximize

\[
z \mathbb{E}[\theta] + (1 - z) \left[ \mu \theta_H + (1 - \mu) \theta_L \right] + z \cdot c + h(z)
\]

by controlling \( z \). From the first-order condition \((\pi - \mu) \Delta \theta + c + h'(z) = 0\), we obtain the optimal choice of \( z \):

\[
z(\mu; c) = h'^{-1}( (\mu - \pi) \Delta \theta - c ).
\]

Its partial derivatives are \( \partial z / \partial \mu = \Delta \theta / h''(z(\mu; c)) < 0 \) and \( \partial z / \partial c = -1 / h''(z(\mu; c)) > 0 \); naturally, an improvement in the belief softens the board’s retention policy and increased aggressiveness increases the probability of CEO replacement.

We then formulate the net mimicry value \( V = U^m - U \) from the truth-telling payoff \( U \) and mimicry payoff \( U^m \). Now that the board does not replace a truth-telling low-type CEO with certainty, the truth-telling payoff is no longer independent of \( c \). The mimicry payoff is

\[
U = \theta_L + (1 - z(0; c)) \chi,
\]

which is no longer independent of \( c \). The mimicry payoff is

\[
U^m = (\theta_L - d) + \chi - \chi \int_0^\infty z(\mu(y, \sigma), c) f(y - \theta_L + d) dy.
\]
The net mimicry value $V = U^m - U$ has a negative partial derivative with respect to $\sigma$,

$$\frac{\partial V}{\partial \sigma} = -\chi \int_{-\infty}^{\infty} \frac{\partial z}{\partial \mu} \frac{\partial \mu}{\partial \sigma} f(y - \theta_L + d) \, dy < 0,$$

but the partial derivative with respect to $c$ is ambiguous:\footnote{We know $z(\mu(y, \sigma); c) < z(0; c)$, but this does not imply $h''(z(\mu(y, \sigma); c)) < h''(z(0; c))$ because $h''(z)$ may have a decreasing part. For example, when $h(z) = \log(z - z^2)$, $h''(z)$ is decreasing on $[1/2, 1]$ because $h'''(z) = 2[x^{-3} - (1 - x)^{-3}]$ is negative for $z > 1/2$.}

$$\frac{\partial V}{\partial c} = \int_{-\infty}^{\infty} \frac{\chi}{h''(z(\mu(y, \sigma); c))} f(y - \theta_L + d) \, dy - \frac{\chi}{h''(z(0; c))} = \frac{\partial U^m}{\partial c} (\theta < 0) - \frac{\partial U}{\partial c} (\theta < 0).$$

By repeating the argument of Appendix 6.1, we find that we cannot determine the sign of $\partial \sigma^*/\partial c$, either.

This ambiguity emerges from the fact that the structure described in Table 1 does not hold in this setup. Because we cannot eliminate the effect of $c$ from $U$, we cannot determine the sign of $\partial V/\partial c = \partial U^m/\partial c - \partial U/\partial c$ unlike in our baseline model, in which $\partial V_{\text{base}}/\partial c = \partial U^m_{\text{base}}/\partial c$ is unambiguously negative. From this argument, we learn that the binary retention decision contributes to simplify the analysis because the simple structure of Table 1 may not be achieved with a continuous decision variable, as exemplified above.

However, we note that the structure described in Table 1 fails to hold in this setup only because we forcefully eliminated the corner solution $z = 1$ by setting $B(z, c) = -\infty$ when $z = 1$. Even when the function $B(z, c)$ is quadratic (e.g., $B(z, c) = cz^2$) or concave (e.g., $B(z, c) = cz^\alpha$ with $\alpha \in (0, 1]$) in $z$, as long as the function $B(z, c)$ does not have an extremely negative value at $z = 1$, the corner solution $z = 1$ easily emerges after truthful reporting and the structure of Table 1 is recovered. We thus conclude that the binariness itself does not seem to be one of the crucial assumptions in the analysis of our baseline model, even though it plays a certain role to simplify the analysis through achieving the simple structure described in Table 1.

### E Proofs for Appendix C

#### E.1 Proof of Proposition 2

When $\theta_1 \equiv E[\theta] + c < \bar{\theta}$, the board unconditionally retains the CEO. In contrast, the board always replace the CEO if $\theta_1 > \bar{\theta}$. In either case, the CEO has no incentive to misreport her type and choose a truthful message as the unique optimal choice.

#### E.2 Proof of Proposition 3

We prove the claim in a series of lemmas.
E.2.1 Proof of Lemma 8

Let $\beta$ denote the posterior distribution after observing $\hat{\theta}$. The expected value of type $\theta$ after observing report $\hat{\theta}$ and output $y$ is

$$M(y) = \frac{\int_{[\hat{\theta},1]} \theta f(y - \theta + d) \, d\beta + \int_{(\hat{\theta},\beta)} \theta f(y - \theta + d \cdot 1_{\{\theta \neq \hat{\theta}\}}) \, d\beta}{\int_{\beta} f(y - \theta + d \cdot 1_{\{\theta \neq \hat{\theta}\}}) \, d\beta}.$$  

The sign of $M(y) - \theta_1$ is identical to that of

$$L(y) = \frac{\int_{\hat{\theta}} f(y - \theta + d \cdot 1_{\{\theta \neq \hat{\theta}\}}) \, d\beta}{f(y - \theta_1 + d)} \cdot (M(y) - \theta_1)$$

$$= \int_{[\hat{\theta},1]} (\theta_1 - \theta) \frac{f(y - \theta + d)}{f(y - \theta_1 + d)} \, d\beta - \int_{(\hat{\theta},\beta)} (\theta_1 - \theta) \frac{f(y - \theta + d \cdot 1_{\{\theta \neq \hat{\theta}\}})}{f(y - \theta_1 + d)} \, d\beta.$$  

Unless the posterior $\beta$ assigns probability 1 on type $\theta_1$, the function $L$ is continuous and decreasing in $y$ due to the monotone likelihood ratio property: the first integral is decreasing and the second is increasing. Therefore, it is optimal for the board to replace the manager when $y < L^{-1}(0)$ and to retain her when $y > L^{-1}(0)$. That is, a cutoff rule with $k = L^{-1}(0)$ is optimal. Here, $L^{-1}(0)$ is well-defined after continuously extending the domain of $L$ to $[-\infty, +\infty]$.

E.2.2 Proof of Lemma 9

The manager’s payoff depends only on the cutoff $k$ when she misreports her type. Therefore, she chooses a report with the minimum cutoff and no type uses any report with a higher cutoff.

E.2.3 Proof of Lemmas 10 and 11

We repeatedly apply the following lemma in this proof.

Lemma 12. Let $k \in [-\infty, +\infty]$ and $z : \mathbb{R} \to [0, 1]$ be a (measurable) retention policy. When $t > t'$, the following four implications hold:

$$\int_k^\infty f(y - t') \, dy \leq \int_k^\infty z(y) f(y - t') \, dy \Rightarrow \int_k^\infty f(y - t) \, dy \geq \int_{-\infty}^\infty z(y) f(y - t) \, dy$$

$$\int_k^\infty f(y - t) \, dy \geq \int_k^\infty z(y) f(y - t) \, dy \Rightarrow \int_k^\infty f(y - t') \, dy \leq \int_{-\infty}^\infty z(y) f(y - t') \, dy$$

$$\int_{-\infty}^k f(y - t) \, dy \geq \int_{-\infty}^k z(y) f(y - t) \, dy \Rightarrow \int_{-\infty}^k f(y - t') \, dy \geq \int_{-\infty}^k z(y) f(y - t') \, dy$$

$$\int_{-\infty}^k f(y - t') \, dy \leq \int_{-\infty}^k z(y) f(y - t') \, dy \Rightarrow \int_{-\infty}^k f(y - t) \, dy \leq \int_{-\infty}^k z(y) f(y - t) \, dy.$$
Moreover, the inequalities in the first two consequents are strict if \( z(y) \neq 1_{\{y > k\}} \) on a set with a positive Lebesgue measure. The inequalities in the last two consequents are strict if \( z(y) \neq 1_{\{y < k\}} \) on a set with a positive Lebesgue measure.

**Proof.** The first two implications follows from

\[
\int_k^\infty f(y - t') \, dy - \int_{-\infty}^\infty z(y) f(y - t') \, dy \\
= f(k - t') \left\{ \int_k^\infty (1 - z(y)) \frac{f(y - t')}{f(k - t')} \, dy - \int_{-\infty}^k z(y) \frac{f(y - t')}{f(k - t')} \, dy \right\} \\
\leq f(k - t') \left\{ \int_k^\infty (1 - z(y)) \frac{f(y - t)}{f(k - t)} \, dy - \int_{-\infty}^k z(y) \frac{f(y - t)}{f(k - t)} \, dy \right\} \\
= \frac{f(k - t')}{f(k - t)} \left\{ \int_k^\infty f(y - t) \, dy - \int_{-\infty}^k z(y) f(y - t) \, dy \right\}.
\]

Here, the inequality is due to the monotone likelihood ration property. Similarly,

\[
\int_{-\infty}^k f(y - t) \, dy - \int_{-\infty}^\infty z(y) f(y - t) \, dy \\
\geq \frac{f(k - t)}{f(k - t')} \left\{ \int_{-\infty}^k f(y - t') \, dy - \int_{-\infty}^\infty z(y) f(y - t') \, dy \right\}.
\]

proves the latter half. In either case, the inequality is strict when the condition in the statement is satisfied.

The following lemma constitutes an essential part of this proof.

**Lemma 13.** Consider an equilibrium that survives the D1 criterion. If message \( \theta \in (\theta_1, \theta) \) is out of equilibrium, then the board assigns probability 1 on types above \( \theta_1 \) after this message. Consequently, no message above \( \theta_1 \) is out of equilibrium.

**Proof.** Let \( \hat{\theta} \) be a message that type \( \theta \) uses in equilibrium and let \( \hat{z}(y) \) be the equilibrium retention policy for message \( \hat{\theta} \).

We first prune the possibility that type \( \theta' \in (\theta, \theta_1] \setminus \{\hat{\theta}\} \) chooses message \( \theta \). Suppose that the type \( \theta' \) weakly prefers message \( \theta \) to \( \hat{\theta} \) if the board uses a cutoff rule with cutoff \( k \) for message \( \theta \). Since type \( \theta' \) is neither \( \theta \) nor \( \hat{\theta} \), we have

\[
\int_k^\infty f(y - (\theta' - d)) \, dy \geq \int_{-\infty}^\infty \hat{z}(y) f(y - (\theta' - d)) \, dy
\]

and thus, by Lemma 12,

\[
\int_k^\infty f(y - \theta) \, dy \geq \int_{-\infty}^\infty \hat{z}(y) f(y - \theta) \, dy.
\]
That is, type $\theta$ has a higher retention rate with the truthful message $\theta$ than the equilibrium message $\hat{\theta}$. Since type $\theta$ gains $d$ in addition by truth-telling, this type strictly prefers the truthful message. Therefore, after observing message $\theta$, the board assigns no probability on the set $(\theta, \theta_1) \setminus \{\hat{\theta}\}$.

We then consider type $\hat{\theta}$. Once again suppose the board uses cutoff $k$ for message $\theta$. The type $\hat{\theta}$ weakly prefers message $\theta$ to $\hat{\theta}$ only if

$$\int_{-\infty}^{\infty} \hat{\theta}(y - (\hat{\theta} - d)) + \hat{\theta}h(y) f(y - \hat{\theta}) \, dy > \int_{-\infty}^{\hat{\theta}} \hat{\theta}h(y - \hat{\theta}) f(y - \hat{\theta}) \, dy. \quad (E.1)$$

First suppose $\theta - d \geq \hat{\theta}$. In this case, define $\hat{k}$ by

$$\int_{-\infty}^{\infty} \hat{\theta}(y) f(y - \hat{\theta}) \, dy = \int_{-\infty}^{\hat{k}} \hat{\theta}h(y - \hat{\theta}) \, dy. \quad (E.2)$$

By combining (E.1) and (E.2), we obtain $F(\hat{\theta} - d - \hat{k}) > F(\hat{\theta} - \hat{\theta})$, or equivalently, $\hat{k} < 2\hat{\theta} - d - k$. Also, from (E.2), by Lemma 12,

$$\int_{-\infty}^{\infty} \hat{\theta}(y)(y - \theta) \, dy \leq \int_{-\infty}^{\hat{k}} \hat{\theta}h(y - \hat{\theta}) \, dy = F(\hat{k} - \theta + d)$$

$$< F(2\hat{\theta} - \theta - k) < F(\theta - k) = \int_{-\infty}^{\infty} f(y - \theta) \, dy.$$

That is, type $\theta$ have a higher retention rate with the truthful message than message $\hat{\theta}$. Therefore, in this case, we prune the possibility that type $\hat{\theta}$ chooses message $\theta$.

Now suppose $\theta - d < \hat{\theta}$. This time, we define $\hat{k}$ by

$$\int_{-\infty}^{\infty} \hat{\theta}(y) f(y - \hat{\theta}) \, dy = \int_{-\infty}^{\hat{k}} \hat{\theta}h(y - \hat{\theta}) \, dy. \quad (E.3)$$

From (E.1) and (E.3), we have $\hat{\theta} - d - k > \hat{\theta} - \hat{k}$ and thus $\hat{k} > k + d$. By Lemma 12, equation (E.3) implies

$$\int_{-\infty}^{\infty} \hat{\theta}(y)(y - \theta) \, dy \leq \int_{-\infty}^{\hat{k}} \hat{\theta}h(y - \hat{\theta}) \, dy = F(\hat{k} - \theta + d)$$

$$< F(\hat{\theta} - d - (k - d)) < F(\theta - k) = \int_{-\infty}^{\infty} f(y - \theta) \, dy.$$

Hence, again, type $\theta$ strictly prefers the truthful message to the equilibrium message. In either case, the board assigns no probability on $(\hat{\theta}, \theta_1)$ after observing message $\theta$ in any D1 equilibrium.

To show the second part of the statement, suppose $\hat{\theta}$ is out of equilibrium. By the first part of this lemma, the board optimally retains the manager for sure after observing that message. If this is the case, the type $\hat{\theta}$ should choose the truthful message $\hat{\theta}$, which contradicts the assumption that message $\hat{\theta}$ is out of equilibrium. 

$\square$
We first show Lemma 11 by combining Lemmas 9 and 13.

**Proof of Lemma 11**

Let \( k(\hat{\theta}) \) denote the cutoff for message \( \hat{\theta} \in (\theta_1, \theta) \). Suppose to the contrary \( k(\hat{\theta}) > k(\hat{\theta}') \) for some \( \hat{\theta}, \hat{\theta}' \in (\theta_1, \theta) \). Then, no type uses message \( \hat{\theta} \) as a misreport by Lemma 9. By Lemma 9, the message \( \hat{\theta} \) needs to be used by type \( \hat{\theta} \). In this case, the board retains the manager for sure, i.e., \( k(\hat{\theta}) = -\infty \). This contradicts with \( k(\hat{\theta}) > k(\hat{\theta}') \).

**Proof of Lemma 10**

First observe that, by Lemma 11, the type \( \theta^* \) prefers the truthful message to any other message above \( \theta_1 \) because all of these messages use the same cutoff \( k_* \). We show that the retention rate for any message \( \hat{\theta} \in (\theta_1, \theta) \) does not exceed the truthful counterpart. That is,

\[
\int_{-\infty}^{\infty} \hat{z}(y) f(y - \hat{\theta} - d) \, dy \leq \int_{k_*}^{\infty} f(y) \, dy, \tag{E.4}
\]

where \( \hat{z}(y) \) is the retention policy for message \( \hat{\theta} \). If this is the case, the message \( \theta^* \) is the unique optimal choice for the type \( \theta^* \).

The condition (E.4) is clearly satisfied when \( k_* = -\infty \). We thus assume \( k_* > -\infty \). In this case, there must be a set of types with positive Lebesgue measure that report some message above \( \theta_1 \) with positive probability because otherwise the board assigns probability 1 on \((\theta_1, \theta)\) and sets \( k(\theta) = -\infty \) for some message \( \theta \) above \( \theta_1 \). At least one of such types differs from \( \hat{\theta} \) and let \( \theta_* \) denote this type. Since type \( \theta_* \) weakly prefers the cutoff \( k_* \) to the retention policy for \( \hat{\theta} \),

\[
\int_{-\infty}^{\infty} \hat{z}(y) f(y - (\theta_* - d)) \, dy \leq \int_{k_*}^{\infty} f(y - (\theta_* - d)) \, dy. \tag{E.5}
\]

By Lemma 12, the inequality (E.5) implies

\[
\int_{-\infty}^{\infty} \hat{z}(y) f(y - (\theta_* - d)) \, dy \leq \int_{k_*}^{\infty} f(y - (\theta_* - d)) \, dy = F(\theta_* - d - k_*) \leq F(\theta^* - k_*).
\]

This is the condition (E.4) and thus the truthful message is uniquely optimal. Therefore, any type above \( \theta_1 \) reports the truthful message with probability 1.

**E.3 Proof of Lemma 6**

First note that, on the equilibrium path, the CEO is replaced if her type is below \( \theta_1 \) and her message is \( \theta_1 \) or below because all the types above \( \theta_1 \) report truthful messages (Lemma 10).\(^{47}\) Thus, for the types below \( \theta_1 \), it is optimal to choose either (a) truthful messages or

\(^{47}\)The entire proof should be interpreted as a measure theoretic statements. For example, we allow some types below \( \theta_1 \) is retained even with messages below \( \theta_1 \) as long as the set of such types has Lebesgue measure 0.

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messages above \( \theta_1 \) accompanied with the uniform cutoff. The threshold \( \theta_2 \) is defined, by \( \text{eq:theta2} \), as the type that makes the CEO indifferent between these two choices. As explained in the text, types more than the threshold \( \theta_2 \) have a higher chance of survival than the indifferent type \( \theta_2 \) and thus prefer (b); types below \( \theta_2 \) prefer (a). Therefore, in equilibrium, types below \( \theta_2 \) (and below \( \theta_1 \)) report truthful messages, whereas types above \( \theta_2 \) (but below \( \theta_1 \)) choose messages above \( \theta_1 \).

### E.4 Proof of Lemma 7

We first show the condition (C.3) is necessary when \( \theta_2 \in (\infty, \theta_1) \). Since the board needs to uniformly choose a single cutoff \( k^* \) for almost every messages \( \hat{\theta} \) above \( \theta_1 \), the uniform cutoff \( k^* \) satisfies the first-order condition

\[
\mathbb{E} \left[ (\theta - \theta_1) f(k_* + d \cdot 1_{\{\theta \neq \hat{\theta}_1\}} - \theta) \Big| \hat{\theta} \right] = 0
\]

for such messages. By the law of total expectation,

\[
0 = \mathbb{E} \left[ 1_{\{\hat{\theta} > \theta_1\}} \cdot (\theta - \theta_1) f(k_* + d \cdot 1_{\{\theta \neq \hat{\theta}\}} - \theta) \right]
\]

\[
= \mathbb{E} \left[ 1_{\{\theta > \theta_1\}} \cdot (\theta - \theta_1) f(k_* - \theta) \right] - \mathbb{E} \left[ 1_{\{\theta \in (\theta_1, \theta_2)\}} \cdot (\theta - \theta_1) f(k_* + d - \theta) \right].
\]

The last two expectations represent the left-hand and right-hand sides of the desired condition (C.3).

The condition (C.3) is equivalent to \( J(k_*; \theta_2) = 0 \), where

\[
J(k_*; \theta_2) = \int_{\theta_1}^{\theta_2} (\theta - \theta_1) \frac{f(k_* - \theta)}{f(k_* - \theta_1)} g(\theta) \, d\theta - \int_{\theta_2}^{\theta_1} (\theta_1 - \theta) \frac{f(k_* + d - \theta)}{f(k_* - \theta_1)} g(\theta) \, d\theta. \tag{E.6}
\]

By the monotone likelihood ratio property, the condition \( J(k_*; \theta_2) = 0 \) has a unique solution \( k_*(\theta_2) \) given \( \theta_2 \). The solution is decreasing in \( \theta_2 \) because \( J(k_*; \theta_2) \) is increasing in \( k_* \) and non-decreasing in \( \theta_2 \). Also, \( k_*(\theta_2) \) is a continuous function because \( J(k_*; \theta_2) \) is jointly continuous (and increasing in \( k_* \)). As \( \theta_2 \) approaches \( \theta_1 \), the solution \( k_*(\theta_2) \) decreases to \( -\infty \) because \( J(k_*; \theta_2) \) converges to a positive value as \( \theta_2 \nearrow \theta_1 \), whenever cutoff \( k_* \) is finite.

### E.5 Proof of Theorem 5

Let \((\theta_2^*, k^*)\) be the unique fixed point. We first construct an equilibrium that survives the D2 (and thus D1) criteria.
Existence

We start by specifying how misreporting types mix their reports. Define $h(\hat{\theta})$ for each $\hat{\theta} \in (\theta_1, \theta)$ by

$$h(\hat{\theta}) = \frac{(\hat{\theta} - \theta_1)f(k^* - \hat{\theta})g(\hat{\theta})}{\int_{\theta_2}^{\theta_1} (\theta_1 - \theta)f(k^* + d - \theta)g(\theta)d\theta}.$$  

This function $h$ works as a density function:

$$\int_{\theta_1}^{\hat{\theta}} h(\hat{\theta}) d\hat{\theta} = \frac{\int_{\theta_1}^{\theta_2} (\hat{\theta} - \theta_1)f(k^* - \hat{\theta})g(\hat{\theta}) d\hat{\theta}}{\int_{\theta_2}^{\theta_1} (\theta_1 - \theta)f(k^* + d - \theta)g(\theta)d\theta}$$  

is equal to 1 because $k^*$ satisfies the condition (C.3).

We consider the following strategies and beliefs. The manager with type $\theta \in [\theta_2, \theta_1]$ uses the density $h(\hat{\theta})$ to randomize the messages $\hat{\theta} \in (\theta_1, \theta)$. The manager with some other type reports a truthful message. After observing message $\hat{\theta} \in (\theta_1, \theta)$, the board calculates a posterior belief by the Bayes’ rule and employs the uniform cutoff $k^*$. After message $\hat{\theta} \in (\theta, \theta_1]$, the board sets cutoff $k = +\infty$ (i.e., replacement for sure).

We briefly discuss the optimality of the strategies. The optimality of $k^*$ follows from the fact that the density function $h(\hat{\theta})$ satisfies the first-order condition (C.2) for all $\hat{\theta} \in (\theta_1, \theta)$. The extreme cutoff $k = +\infty$ is a best response for the lower messages because after these messages the board assigns no probability on types above $\theta_1$. The manager with type above $\theta_1$ chooses truthful messages as a unique optimum to get the highest retention rate and the additional productivity $d$. By the definition of $\theta_2$, the remaining types also choose optimal messages for them.

It remains to show that this belief systems survives the D1 and D2 criteria. Consider a cutoff $k_{\varepsilon} = k^* - \varepsilon$ slightly lower than $k^*$. Suppose that this cutoff $k_{\varepsilon}$ is accompanied with an out-of-equilibrium message $\hat{\theta} \in [\theta_2, \theta_1]$. Then, when $\varepsilon$ is sufficiently small, type $\theta = \hat{\theta}$ strictly prefers the truthful message $\hat{\theta}$ for this type to the equilibrium misreporting, whereas all the other types get worse off with this message than their equilibrium messages. Note that the cutoff $k_{\varepsilon}$ becomes a best response for the board by controlling the belief for this message; when the board assigns probability $p$ on $(\theta_1 + \theta_2)/2$ and $1 - p$ on $(\theta + \theta_1)/2$, we can make any level of cutoff a best response by correctly adjusting $p$. Therefore, neither the D1 or D2 criterion can eliminate such a belief.

D2 Criterion

The necessary conditions for D1 equilibria are already shown by Lemmas 10–6. It remains to show that, in any D2 equilibrium, the board must replace the CEO after (almost) every message $\hat{\theta} \in (\theta_2, \theta_1)$.

We claim that the board never assign probability on types above $\hat{\theta}$ for all out-of-equilibrium messages $\hat{\theta}$. To this end, we assume that the manager with type $\theta^* > \hat{\theta}$ weakly prefers the message $\hat{\theta}$ with a retention policy $z$ to the equilibrium message for type $\theta^*$ and show that some type $\theta_z$ strictly prefers the message $\hat{\theta}$ to the equilibrium message for type $\theta_z$.  

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First consider the case that the retention policy $z$ differs from the cutoff rule with cutoff $k^*$ in a measure-theoretic sense; i.e., $\{y : z(y) \neq 1_{\{y>k^*\}}\}$ has a positive Lebesgue measure. The manager with type $\theta^* > \hat{\theta}$ weakly prefers the message $\hat{\theta}$ with the above retention policy to the equilibrium message only if the retention rate with message $\hat{\theta}$ and retention policy $z$ is at least as high as with some misreport above $\theta_1$ and cutoff $k^*$:

$$\int_{-\infty}^{\infty} z(y)f(y-(\theta^*-d))\,dy \geq \int_{k^*}^{\infty} f(y-(\theta^*-d))\,dy.$$

Let $\theta_* \in (\theta_2, \hat{\theta})$ be a type that chooses a message above $\theta_1$ and faces cutoff $k^*$ in equilibrium. By Lemma 12, the type $\theta_*$ has a higher chance of retention with message $\hat{\theta}$ and retention policy $z$ than with the equilibrium message and the cutoff $k^*$. That is, when such retention policy $z$ is accompanied with message $\hat{\theta}$, type $\theta_*$ strictly prefers the message $\hat{\theta}$ to the equilibrium message for this type whenever type $\theta^*$ weakly prefers the message $\hat{\theta}$ to the equilibrium message for type $\theta$.

Now consider the case that the retention policy $z$ is identical to the cutoff rule with $k^*$. (We need the D2 criterion, instead of D1, just for this part.) In this case, the type $\hat{\theta}$ prefers the truthful message $\theta$ to its equilibrium choice, accompanied with the uniform cutoff $k^*$, whereas the type $\hat{\theta}$ is indifferent between $\theta$ and the equilibrium choice (or prefers the latter). That is, even when $z$ is identical to the cutoff rule with $k^*$, we can find a type (in this case, type $\hat{\theta}$) strictly prefer message $\hat{\theta}$ to the equilibrium message.

We have shown that whenever type $\theta^* > \hat{\theta}$ weakly prefers message $\theta$ to its equilibrium message, there exists some type that strictly prefers message $\hat{\theta}$ to the equilibrium message for that type. Therefore, after observing $\hat{\theta}$, the board assigns probability 1 on types below $\hat{\theta}$ ($< \theta_1$) and replaces the CEO regardless of the realization of $y$.

### E.6 Proof of Proposition 5

Case (iv) is obvious. We focus on cases (i)–(iii); i.e., $|c| < \Delta/2$. We first provide a necessary condition for equilibrium. Let $\theta_* = \max\{\theta_2, \theta_3\}$ and $\theta^* = \min(\theta, k + q)$.

**Lemma 14.** Suppose $|c| < \Delta/2$. In equilibrium, $\theta_1 - \theta_* = \theta^* - \theta_1$ and $k \in [\theta_1 - q, \theta_* - d + q]$.

**Proof.** Given $\theta_2 < \theta_1$, the objective of the board is to maximize

$$B(k; \theta_2) = \int_{\theta_1}^{\theta} (\theta - \theta_1)F(k-\theta)\,d\theta + \int_{\theta_*}^{\theta_1} (\theta - \theta_1)F(k+d-\theta)\,d\theta.$$  \hspace{1cm} (E.7)

---

48 The type $\hat{\theta}$ does not necessarily eliminate all the other types through the D1 criterion. Consider a type $\theta \in (\theta_2, \theta + d/2)$. We can easily show that when the board uses a reversed cutoff rule with a cutoff level $k$ that makes type $\hat{\theta}$ indifferent (i.e., $F(k-\theta) = F(\theta-k^*)$), the retention rate for type $\hat{\theta}$ is lower with the truthful message $\theta$ than with the equilibrium misreports (i.e., $F(k-\theta - d) < F(k-\theta - d)$). In particular, the type $\theta$ can be more than $\theta_1$ when $\hat{\theta}$ is close enough to $\theta_1$; that is, the D1 criterion may not be able to eliminate some types above $\theta_1$.

49 The type $\theta$ may get a better deal than cutoff $k^*$ in equilibrium because the posterior belief is indeterminate on a set of measure 0. If it is the case, we do not need the argument for the D2 criterion; the D1 criterion suffices. In general, of course, we need the D2 criterion to obtain the desired result.
The function $B(k; \theta_2)$ is zero when no type survives (i.e., $k \geq \tilde{\theta} + q$). Also, $B(k; \theta_2)$ is positive when $k \in [\theta_1 - d + q, \tilde{\theta} + q)$ because no type below $\theta_1$ survives. Among these values of $k$, the lowest value $k = \theta_1 - d + q$ gives the highest rate of retention and the highest value of $B(k)$ within the interval. Thus, the values of $k > \theta_1 - d + q$ are all suboptimal. Similarly, it is suboptimal to choose $k < \theta_1 - q$ because the retention rate of types below $\theta_1$ increases as $k$ decreases on that region. Therefore, we can focus on $k \in I = [\theta_1 - q, \theta_1 - d + q]$.

We further partition the interval $I$ into three intervals: $I_1 = [\theta_1 - q, \tilde{\theta} - q]$, $I_2 = (\tilde{\theta} - q, \theta_2 - d + q)$, and $I_3 = [\theta_2 - d + q, \theta_1 - d + q]$. The second interval $I_2$ is nonempty because of the second regularity condition (C.5). Actually, $k \in I_3$ never occurs in equilibrium because $k \in I_3$ implies that the threshold type $\theta_2$ has no chance of survival. This contradicts the definition of $\theta_2$: the manager with type $\theta_2$ needs to be indifferent between truth-telling and misreporting.

We investigate the optimality condition for $k$. If $k \in I_1$, then $\theta^* = k + q$ and the function $B(k; \theta_2)$ becomes

$$B(k; \theta_2) = \int_{\theta_1}^{\theta^*} (\theta - \theta_1) \cdot 1 \, d\theta + \int_{\theta_1}^{\theta^*} (\theta - \theta_1) \frac{k - \theta + q}{2q} \, d\theta$$

$$+ \int_{\theta_1}^{\theta^*} (\theta - \theta_1) \frac{k + d - \theta + q}{2q} \, d\theta. \quad (E.8)$$

On the second interval $I_2$, we obtain $\theta^* = \tilde{\theta}$ and

$$B(k; \theta_2) = \int_{\theta_1}^{\theta^*} (\theta - \theta_1) \frac{k - \theta + q}{2q} \, d\theta + \int_{\theta_1}^{\theta^*} (\theta - \theta_1) \frac{k + d - \theta + q}{2q} \, d\theta,$$  
and $(E.10)$. In either case,

$$\frac{\partial B}{\partial k} = \frac{1}{4q} \left\{ (\theta_1 - \theta^*)^2 - (\theta^* - \theta_1)^2 \right\} \quad (E.10)$$

and thus in equilibrium $(\theta_1 - \theta^*)^2 = (\theta^* - \theta_1)^2$, or equivalently, $\theta_1 - \theta^* = \theta^* - \theta_1$ must be satisfied.

Consider case (i) of this proposition. By Lemma 14, $\theta^* = \theta_2 > \theta$ must hold because otherwise $\theta^* - \theta_1 < \Delta/2 < \theta_1 - \tilde{\theta} = \theta_1 - \theta^*$. Also, $\theta^* = \tilde{\theta} < k + q$ because the second regularity condition (C.5) implies

$$\theta - \theta_{2,0} > \tilde{\theta} - q \quad (E.11)$$

and thus $k + q \geq \theta + q - \theta_{2,0} > \tilde{\theta}$. Therefore, an equilibrium candidate is uniquely given by the equilibrium condition $\tilde{\theta} - \theta_1 = \theta_1 - \theta_2$. To verify it is indeed an equilibrium, we simply need to confirm $\theta_2 = 2\theta_1 - \tilde{\theta} > \theta$ (by $c > 0$) and $k = 2\theta_1 - \tilde{\theta} - \theta_{2,0} > \tilde{\theta} - q$ (by (E.11)).

We next consider case (ii). In this case, $\theta^* = k - q < \tilde{\theta}$ holds because otherwise $\theta - \theta_1 > \Delta/2 > \theta_1 - \theta^*$. Once again by (E.11), we obtain $\theta_2 = k + \theta_{2,0} < \tilde{\theta} + q + \theta_{2,0} < \tilde{\theta}$ and thus $\theta_2 = \tilde{\theta}$. These conditions uniquely determine an equilibrium: $k = 2\theta_1 - \tilde{\theta} - q$ and $\theta_2 = 2\theta_1 - \tilde{\theta} - q + \theta_{2,0}$. 

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In case (iii), \( \theta^* = \bar{\theta} \) and \( \theta_* = \bar{\theta} \) hold. Suppose otherwise. Then, \( \theta^* - \theta_1 = \theta_1 - \theta_* < \Delta/2 \) and thus \( \theta^* = k + q \) and \( \theta_* = \theta_2 \) hold. However, \( \theta_2 = k + \theta_{2,0} < \bar{\theta} - q + \theta_{2,0} < \bar{\theta} \) by (E.11); a contradiction. In this case, any value of \( k \) works as an equilibrium cutoff as long as \( k + q \geq \bar{\theta} \) and \( k + \theta_{2,0} \leq \bar{\theta} \).

### E.7 Proof of Theorem 6

We first calculate the payoff function for shareholders.

**Lemma 15.** Assume the regularity conditions (C.5) and (C.4). The equilibrium payoff for the shareholder with exogenous \( c \) is

\[
V_A(c) = \frac{1}{\Delta q} \left\{ \frac{2}{3} c^3 - \left( \frac{dA}{4} + \frac{\Delta}{2} \right) c^2 + d\left( \alpha \cdot \frac{\Delta}{2} + q \right) c + v_0 \right\}
\]

when \( c \in (0, \Delta/2) \), and

\[
V_F(c) = \frac{1}{\Delta q} \left\{ -\frac{2}{3} c^3 - \left( \frac{d}{4} + \frac{\Delta}{2} \right) c^2 - dq \cdot c + v_0 \right\}
\]

when \( c \in (-\Delta/2, 0) \), where \( v_0 = \left[ 2\Delta^2 - 3d(8q - \Delta) \right] \Delta/48 \).

**Proof.** First assume \( c \in (0, \Delta/2) \). By Proposition 5, we have \( k = 2\theta_1 - \bar{\theta} - \theta_{2,0}, k < \bar{\theta} - q \), and \( \theta_2 = 2\theta_1 - \bar{\theta} \) in equilibrium. Thus, the payoff for shareholders, multiplied by \( \Delta q \), is

\[
\Delta q V_A = q \left\{ -d(\theta_1 - \theta_2) + \int_{\bar{\theta}}^{\theta_1} (\theta - \mu) F(\theta - k) \ d\theta + \int_{\theta_2}^{\theta_1} (\theta - \mu) F(\theta - d - k) \ d\theta \right\}
\]

\[
= -dq \left( \frac{\Delta}{2} - c \right) + \frac{1}{2} \left[ \mu(k - q)\theta - (k - q + \mu) \frac{\theta^2}{2} + \frac{\theta^3}{3} \right]_{\theta=\mu+c}^{\mu+\Delta/2}
\]

\[
+ \frac{1}{2} \left[ \mu(k + d - q)\theta - (k + d - q + \mu) \frac{\theta^2}{2} + \frac{\theta^3}{3} \right]_{\theta=\mu+2c-\Delta/2}^{\mu+c}
\]

\[
= -dq \left( \frac{\Delta}{2} - c \right) + \left[ c^3 - \frac{2d(1 + \alpha)}{8} \right] - \frac{\Delta^2}{8} c + \frac{\Delta^2}{96} \left\{ 6d(1 + \alpha)d + 5\Delta \right\}
\]

\[
+ \left[ c^3 - \frac{6d\alpha + 3\Delta}{8} \right] c^2 + \frac{\Delta}{8} (4d\alpha + \Delta)c - \frac{\Delta^2}{96} \left\{ 6d\alpha + \Delta \right\}.
\]

We obtain the desired expression by simplifying the above expression.

We next consider the case of \( c \in (-\Delta/2, 0) \). We know that \( k = 2\theta_1 - q - \bar{\theta}, k < \bar{\theta} - q \)
and \( \theta_2 < \theta \) due to Proposition 5. Hence,

\[
\Delta q V_F = q \left\{ -d(\theta_1 - \theta_2) + \int_{\theta_1}^{\theta} (\theta - \mu) d\theta + \int_{\theta}^{k+q} (\theta - \mu) F(\theta - k) d\theta + \int_{\theta}^{\theta_1} (\theta - \mu) F(\theta - d - k) d\theta \right\}
\]

\[
= -dq \left( c + \frac{\Delta}{2} \right) - q \left[ 2c^2 + \Delta c \right] + \left[ -\frac{c^3}{3} + \frac{3}{8} (4q - \Delta) c^2 + \left( \Delta q - \frac{\Delta^2}{8} \right) c + \frac{\Delta^2}{96} (12q - \Delta) \right]
\]

\[
+ \left[ -\frac{c^3}{3} + \frac{1}{8} (4q - \Delta - 2d)c^2 + \frac{\Delta^2}{8} \cdot c + \frac{\Delta^2}{96} (6d + 5\Delta - 12q) \right].
\]

Once again, we obtain the desired expression after a few more calculations. \( \square \)

These two cubic functions \( V_A \) and \( V_F \) may have a local maximum at \( c_A^* \) and \( c_F^* \), respectively. We later see that the locally maximum values (i.e., \( V_A(c_A^*) \) and \( V_F(c_F^*) \)) are proportional to

\[
\Omega(x) = -2(1 + 4x)^3 d^3 + 2(1 + 4x)d^2 \left[ (1 + 4x)\sqrt{D(x)} - 6\Delta + 48q \right]
\]

\[
+ 8d(\Delta - 8q) \left[ \sqrt{D(x)} + 3\Delta \right] + 8\Delta^2 \left[ \sqrt{D(x)} + 2\Delta \right]
\]

when \( x = \alpha \) and 0, respectively, where \( D(x) = d^2(1 + 4x)^2 + 4d\Delta - 32dq + 4\Delta^2 \). Note that \( D(\alpha) \) is positive because

\[
D(\alpha) > 16\alpha^2 d^2 + 8\alpha d^2 + d^2 + \frac{16dq}{2\alpha + 1} - 32dq + \frac{64q^2}{(2\alpha + 1)^2} = \frac{(8\alpha^2 d + 6\alpha d + d - 8q)^2}{(2\alpha + 1)^2} > 0.
\]

(E.12)

The inequality follows from the first regularity condition (C.4), or equivalently, \( \Delta > 4q/(1 + 2\alpha) \).

**Lemma 16.** If \( D(0) > 0 \) and (C.5) holds, then \( \Omega'(x) > 0 \) for all \( x \in (0, \alpha) \). Also, \( \Omega'(x) > 0 \) for all positive \( x > \frac{4q - \Delta}{2\Delta} \).

**Proof.** The first-order derivative of \( \Omega \) is given by

\[
\frac{1}{24d^2} \Omega'(x) = 16q - 2\Delta - d(1 + 4x)^2 + (1 + 4x)\sqrt{D(x)}.
\]

(E.13)

If \( 16q \geq 2\Delta + d(1 + 4x)^2 \), then (E.13) is positive; this is the case when (C.5) holds. Suppose otherwise. Then, (E.13) is positive if and only if

\[
\lambda(x) = \frac{1}{32} \left\{ (1 + 4x)^2 D(x) - [d(1 + 4x)^2 + 2\Delta - 16q]^2 \right\} = x(1 + 2x)\Delta^2 - 2q(4q - \Delta)
\]

\[
= \frac{1}{32} \left\{ (1 + 4x)^2 D(x) - [d(1 + 4x)^2 + 2\Delta - 16q]^2 \right\} = x(1 + 2x)\Delta^2 - 2q(4q - \Delta)
\]
Hence, \( q < \Delta \) is an elementary property of cubic functions. Thus, we only need to show that the condition \( c - V \) by Lemma 16 and the second regularity condition (C.5). In either case, \( V \) the domain \( (0, 2\Delta) \). The first-order derivative of \( c \) is positive. We show that the condition (C.4) guarantees that \( c^* \) is a unique local maximizer on the domain \( (0, \Delta/2) \). The root \( c^* \) is always positive because \( D(\alpha) = (dA + 2\Delta)^2 - 16d(\alpha\Delta + 2q) < (dA + 2\Delta)^2 \).

We show that the condition (C.4) guarantees \( c^*_A < \Delta/2 \), or equivalently, \( \sqrt{D(\alpha)} > 2\Delta - dA \). Since (C.4) is equivalent to \( q < \Delta(1 + 2\alpha)/4 \), we obtain

\[
D > d^2 A^2 + 4d\Delta - 32d \cdot \frac{\Delta(1 + 2\alpha)}{4} + 4\Delta^2 = (2\Delta - dA)^2.
\]

Hence, \( \sqrt{D(\alpha)} > \sqrt{(2\Delta - dA)^2} = 2\Delta - dA \) if \( 2\Delta - dA \geq 0 \); otherwise, \( \sqrt{D(\alpha)} > 0 > 2\Delta - dA \). In either case, we obtain \( c^* < \Delta/2 \) and thus the point \( c^*_A \) is a local maximizer on the domain \( (0, \Delta/2) \).

Now we show \( V_A(c^*_A) \) is always positive. After several calculations, we obtain \( V_A(c^*_A) = \frac{1}{768\Delta q} \Omega(\alpha) \). By Lemma 16,

\[
768\Delta q V_A(c^*_A) = \Omega(\alpha) > \Omega \left( \frac{4q - \Delta}{2\Delta} \right)
\]

\[
= 2\Delta^{-3} \left[ 2\Delta^2 - d(8q - \Delta) \right] \left\{ \left[ 2\Delta^2 - d(8q - \Delta) \right] + \left[ 2\Delta^2 - d(8q - \Delta) \right] \right\} \geq 0.
\]

because \( \alpha > \frac{4q - \Delta}{2\Delta} \) by (C.4).

Lastly, observe that the local maximum attained at \( c^*_A \) is indeed a global maximum on \( (0, \Delta/2) \) because \( V'_A(0) > 0 \) and \( V_A(\Delta/2) = 0 \).

To complete the proof of Theorem 6, we show \( V_F \) never exceeds \( V_A(c^*) \). First note if \( V_F \) does not have a local maximizer on its domain \((-\Delta/2, 0)\), then \( V_F \) is negative because \( V_F \) is a cubic function with \( V_F(0) < 0 \) and a negative coefficient on \( c^3 \). Suppose a local maximizer \( c^*_F \) exists. Then, it must be the larger root of \( V'_F(c) = 0 \); that is, \( c^*_F = (\sqrt{D(0)} - d - 2\Delta)/8 \). The local maximum is given by \( V_F(c^*_F) = \frac{1}{768\Delta q} \Omega(0) \), which is less than \( V_A(c^*_A) = \frac{1}{768\Delta q} \Omega(\alpha) \) by Lemma 16 and the second regularity condition (C.5). In either case, \( V_F \) cannot exceed \( V_A(c^*_A) \) and it is therefore optimal for shareholders to choose \( c = c^*_A \).
E.8 Proof of Proposition 6

The first-order derivatives of \(c^*_A\) are

\[
\frac{\partial c^*_A}{\partial d} = \frac{1}{8\sqrt{D}} \left\{ (1 + 4\alpha)\sqrt{D} + [16q - d(1 + 4\alpha)^2 - 2\Delta] \right\}
\]

\[
\frac{\partial c^*_A}{\partial \Delta} = \frac{1}{4\sqrt{D}} \left\{ \sqrt{D} - (d + 2\Delta) \right\}
\]

\[
\frac{\partial c^*_A}{\partial \chi} = \frac{\alpha^2 d}{4q\sqrt{D}} \left\{ d(1 + 4\alpha) - \sqrt{D} \right\}
\]

\[
\frac{\partial c^*_A}{\partial q} = \frac{d}{2q\sqrt{D}} \left\{ \alpha\sqrt{D} + [4q - \alpha(1 + 4\alpha)d] \right\}
\]

where \(\alpha = 2q/\chi\) and \(D = d^2(1 + 4\alpha)^2 + 4d\Delta - 32dq + 4\Delta^2 (> 0)\). First, \(\partial c^*_A/\partial d > 0\) because the second regularity condition (C.5) implies \(d \leq \frac{2(8q-\Delta)}{(1+4\alpha)^2}\) and thus

\[
16q - d(1 + 4\alpha)^2 - 2\Delta \geq 16q - \frac{2(8q - \Delta)}{(1 + 4\alpha)^2} \cdot (1 + 4\alpha)^2 - 2\Delta = 0.
\]

Second, \(\partial c^*_A/\partial \Delta < 0\) because

\[
D - (d + 2\Delta)^2 = 8d \left\{ \alpha(1 + 2\alpha)d - 4q \right\} \geq 8d \left\{ \alpha(1 + 2\alpha) \cdot \frac{2(8q - \Delta)}{(1 + 4\alpha)^2} - 4q \right\}
\]

\[
= -\frac{16d}{(1 + 4\alpha)^2} \left\{ \alpha(1 + 2\alpha)\Delta + 2(1 + 4\alpha + 8\alpha^2)q \right\} < 0
\]

by the second regularity condition (C.5). Third, \(\partial c^*_A/\partial \chi\) has the opposite sign of \(d(\Delta - 8q) + \Delta^2\) because \(D = \{d(1 + 4\alpha)\}^2 - 4\{d(\Delta - 8q) + \Delta^2\}\).

Lastly, we show \(\partial c^*_A/\partial q > 0\). This is obvious when \(4q \geq \alpha(1 + 4\alpha)d\). Suppose otherwise; i.e., \(d > \frac{4q}{\alpha(1 + 4\alpha)}\). Then, a necessary and sufficient condition for \(\partial c^*_A/\partial q > 0\) is

\[
\alpha^2 D - [\alpha(1 + 4\alpha)d - 4q]^2 = 4(\alpha\Delta + 2q) \left\{ \alpha(d + \Delta) - 2q \right\}
\]

is positive. This condition is true because

\[
\alpha(d + \Delta) - 2q > \alpha \left\{ \frac{4q}{\alpha(1 + 4\alpha)} + \frac{4q}{2\alpha + 1} \right\} - 2q = \frac{2q}{(1 + 2\alpha)(1 + 4\alpha)} > 0
\]

due to the supposition \(d > \frac{4q}{\alpha(1 + 4\alpha)}\) and the first regularity condition (C.4).