Computing large market equilibria using abstractions

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Abstract

Computing market equilibria is an important practical problem for market design (e.g. fair division, item allocation). However, computing equilibria requires large amounts of information (e.g. all valuations for all buyers for all items) and compute power. We consider ameliorating these issues by applying a method used for solving complex games: constructing a coarsened abstraction of a given market, solving for the equilibrium in the abstraction, and lifting the prices and allocations back to the original market. We show how to bound important quantities such as regret, envy, Nash social welfare, Pareto optimality, and maximin share when the abstracted prices and allocations are used in place of the real equilibrium. We then study two abstraction methods of interest for practitioners: 1) filling in unknown valuations using techniques from matrix completion, 2) reducing the problem size by aggregating groups of buyers/items into smaller numbers of representative buyers/items and solving for equilibrium in this coarsened market. We find that in real data allocations/prices that are relatively close to equilibria can be computed from even very coarse abstractions.

1 Introduction

The problem of solving for the equilibrium prices and allocation of a market economy has large informational and computational requirements (Hayek [1945]). In this work we apply the idea of abstraction to market equilibrium computation. We construct a simplified model of a market (aka. an abstraction), solve for the equilibrium in the abstraction, and then lift the answers back to the original market. We derive analytic bounds for the error in the computed equilibrium as a function of abstraction quality, describe two methods of abstraction that can be useful in practical problems, and evaluate our approach on real datasets.

Computing optimal allocations subject to constraints has been a problem of interest since the inception of modern economic theory. Early applications included trying to use linear programming to plan the entire Soviet economy (Kantorovich [1960], [1975]). Modern market designers, with slightly less grand visions, also use centralized algorithms to construct allocations with various properties such as incentive compatibility, envy free-ness, stability, efficiency, and notions of fairness (Roth [2002], Klemperer [2018]).

The computation of market equilibrium is used as a component of a powerful allocation algorithm: competitive equilibrium from equal incomes, CEEI (Varian et al. [1974]). Individuals are endowed with a budget of pseudo currency, give their valuations for items, an equilibrium of the resulting market is computed, and individuals receive the allocation from this equilibrium. This mechanism is known to have good allocative properties including that it is approximately incentive compatible in the large (Budish [2011]) and produces allocations which maximize the product of utilities (a.k.a Nash social welfare), a criterion that “exhibits an elusive combination of fairness and efficiency properties”
We will focus on the canonical case of Fisher markets. In Fisher markets we have a set of divisible items to be allocated and a set of individuals who may receive the items. Individuals have a total budget of money and valuations for each item. Formally, a market equilibrium consists of a price for each item and an allocation such that 1) individuals cannot improve their utility by using their budgets to purchase a different set of items (given the prices) and 2) the total demand for each item (i.e. the sum of the solution of each individual’s maximization problem of allocating their budget) is equal to the supply.

The equilibrium in Fisher markets that maximizes Nash social welfare can be computed by solving the *Eisenberg-Gale (EG) convex program* [Eisenberg and Gale, 1959]. In addition to being useful for fair division (Caragiannis et al., 2016) the EG solution has strong connections to budget smoothing in auction markets (Conitzer et al., 2018a). More generally Fisher markets and their equilibria can be related to methods for budget smoothing both in first- and second-price auction markets (Conitzer et al., 2018b,a).

Scaling market equilibrium computation to larger markets requires solving two major challenges. First, one needs access to large amounts of information (e.g. every person’s valuation for every item or item combination). Second, one needs access to enough compute power to solve the related convex or mixed integer program which, even if it is theoretically solvable in weakly polynomial time (Nisan et al., 2007), becomes unwieldy for large (e.g. thousands or millions of buyers) markets.

In this paper we consider solving for equilibria of large markets by introducing the notion of *market abstractions*. We use the information we have to create an abstraction (a simplified model of the market), solve for the equilibrium in the abstraction, and then project the answers to the original market. We provide a general set of results that bound various quantities (regret, envy, Pareto optimality, Nash social welfare) of buyers if we use the allocation/prices derived from the abstraction rather than the true equilibrium.

We then turn to practice and focus on two abstraction methods of particular interest. The first is using techniques from matrix completion to infer valuations for person/item pairs that we may not have access to. We refer to this as *low rank markets*. The second is reducing the size of the market (and thus computational burden) by replacing groups of buyers (or items) with a *representative buyer* or *representative item*, solving for equilibrium in the representative case and then splitting the allocation of each representative among the individuals/items it represents. We refer to this as *representative market equilibrium*. We show that this abstraction reduces computational complexity and can be efficiently parallelized. We show two ways of performing the lift to the original market, a proportional and a recursive version, and discuss the tradeoffs of each one.

We apply these abstraction methods, which can be used together, to real datasets including a novel one which we have collected and evaluate the quality of solutions with various levels of abstraction. We find that the equilibria found even in very coarse abstractions have quite decent properties.

### 2 Related Work

The use of equilibrium assumptions to make estimates of deep ‘structural’ parameters is of interest in both applied micro and macroeconomics (Berry et al., 1995; Ljungqvist and Sargent, 2018). Often the lack of individual-level data and/or compute power requires the use of representative agent (i.e. a single consumer that represents all individuals in the economy). To get around this issue, analysts typically make (strong) assumptions which imply that the equilibrium prices/aggregate decisions/some structural parameters computed using a representative-agent stand-in are equivalent to the ones that would be derived from a model which includes all individual agents. In our work we are specifically interested in the situation where these assumptions are not true and the lifted answer from the abstraction does not yield the true equilibrium prices/allocation. There is increasing interest in using heterogenous agent models in applied economic modeling (Hommes, 2006) and expanding our results to these cases is an interesting direction for future work.

Abstraction is an idea often used in the context of games. In two-player, zero-sum games (e.g. poker) a popular goal is to compute a Nash equilibrium strategy during training time and use it in actual play.
against opponents. In large games, however, it is impractical to solve for the Nash equilibrium of the original game and practitioners solve for the Nash equilibrium of an abstraction and then lift it to be a strategy for the original game. Abstractions often use heuristics and are hand tuned (Gilpin et al., 2007; Waugh et al., 2009; Ganzfried and Sandholm, 2014; Brown et al., 2015; Brown and Sandholm, 2018) but more recently have begun to be constructed automatically using function approximation (e.g. deep learning) (Moravčík et al., 2017; Brown et al., 2018). In games, unlike in markets, the relationship between the quality of abstraction and the quality of the lifted strategy in the original game has been heavily studied (Lanctot et al., 2012; Kroer and Sandholm, 2014, 2016, 2018).

There is some literature on abstraction in non-market-based allocation problems (Walsh et al., 2010; Peng and Sandholm, 2016; Lu and Boutilier, 2015). The scalability problem faced in the allocation setting is similar to ours, but because the underlying optimization problem is very different (i.e. the maximization of allocative efficiency rather than a market equilibrium) the abstraction methods and results in these papers are quite different in character from ours.

Finally there is a large literature on computing market equilibria in Fisher markets using convex programming (Eisenberg and Gale, 1959; Shmyrev, 2009) or gradient-based methods (Birnbaum et al., 2011; Nesterov and Shikhman, 2018). There is also work extending the EG program to new settings such as quasi-linear utilities and indivisible items (Cole et al., 2017; Cole and Gkatzelis, 2018; Caragiannis et al., 2016). Our paper complements this existing work as our results are agnostic to the algorithm used for equilibrium computation; any of these algorithms can be employed for computing a market equilibrium in conjunction with our market abstraction model.

3 Market Equilibrium in Fisher Markets

In this paper we focus on the canonical Fisher market setting with linear utilities. We have a set of \( n \) buyers and a set of \( m \) goods (items). We assume that items are divisible and denote by \( X \) a matrix of allocations of items to buyers where \( x_{ij} \) refers to the share of item \( j \) allocated to buyer \( i \). Each item \( j \) has a supply of \( s_j \). We say an allocation is a full allocation if the sum of each column \( j \) of \( X \) is \( s_j \).

Each buyer \( i \) is endowed with a budget \( B_i \). We denote by \( V \in \mathbb{R}_{n,m}^+ \) the matrix of values with \( v_{ij} \) being the value of buyer \( i \) for item \( j \) and \( v_i \) the vector of values for buyer \( i \). For a given allocation \( X \) we assume that total values are additive and linear i.e.

\[
 u_i(x_i) = v_i \cdot x_i = \sum_j v_{ij} x_{ij}
\]

**Definition 1.** Given prices \( p \in \mathbb{R}_m^+ \) for the goods, a demand for buyer \( i \) is

\[
 d_i(p) = \{ \arg \max_{x : x \cdot p \leq B_i} v_i \cdot x_i \}.
\]

The demand can be set valued but the maximum reachable utility given a price vector is unique.

**Definition 2.** A set of prices and an allocation \((p, X)\) is a market equilibrium if \( x_i \in d_i(p) \) and \( X \) is a full allocation.

An equilibrium is computable using the Eisenberg-Gale (EG) convex program (Eisenberg and Gale, 1959) whose solution has an elegant structure: for every good that an individual is receiving in their equilibrium allocation they receive equal ‘bang per buck’. Formally, in equilibrium \((p^*, X^*)\)

\[
 \text{if } x^*_{ij}, x^*_{ij'} > 0 \text{ then } \frac{v_{ij}}{p^*_j} = \frac{v_{ij'}}{p^*_j'}
\]

In EG equilibrium prices and per-buyer utilities are unique (there may be multiple equilibrium allocations, for example, if buyers view 2 goods as perfectly interchangeable). We refer to this market equilibrium as the EG equilibrium.

The structure of the EG equilibrium makes CEEI in Fisher markets attractive for centralized division algorithms since this equilibrium maximizes the geometric mean of individual utilities weighted by budgets - i.e. the Nash social welfare when budgets are 1 (Varian et al., 1974; Caragiannis et al., 2016). For this reason the solution is also referred to as the max Nash welfare (MNW) allocation.
4 Abstractions of Markets

We are interested in settings where we do not have full access to the valuation matrix $V$. Rather, we have an abstraction of $V$ and a method $L$ to deterministically lift any prices and allocations back up to the original market. We refer to these lifted quantities as $(\hat{p}, \hat{X})$. In addition our lift is such that if $(\hat{p}, \hat{X})$ correspond to an equilibrium in the abstraction, then the lift returns a $\hat{V}$ of the same size as the original market such that $(\hat{p}, \hat{X})$ form an equilibrium in $\hat{V}$.

This motivates our main question

**Question 1 (Main Question).** If $(\hat{p}, \hat{X})$ is an equilibrium with respect to $\hat{V}$ what can be said about it relative to the true valuation matrix $V$?

**Definition 3.** We define the abstraction error as $\Delta V = V - \hat{V}$. We use $\Delta v_i$ to denote $v_i - \hat{v}_i$.

We can prove the following simple Lemma:

**Lemma 1.** If $X, X'$ are feasible allocations such that $\hat{v}_i \cdot x_i + \epsilon \geq \hat{v}_i \cdot x'_i$ then $v_i \cdot x_i + \epsilon \geq v_i \cdot x'_i + \Delta v_i \cdot (x_i - x'_i) \geq v_i \cdot x'_i - \|\Delta v_i\|_1$.

This Lemma will give us our main theoretical results:

**Result 1 (Main Results (Informal)).** Properties which are ‘linear’ in prices and allocations can be related linearly to various matrix norms of $\Delta V$. These properties include regret, Nash social welfare, envy, Pareto optimality, and maximin share. Quantifying the error in the abstraction based on norms of $\Delta V$ has a particular advantage: it allows us to apply well-known effective data-driven matrix methods such as low rank approximation and $k$-means clustering. The next section discusses our theoretical results in detail.

5 Approximation Results

We now look at how well properties of market equilibria are maintained in the equilibria derived from abstractions. We relate these quantities to various matrix norms of the error matrix $\Delta V$. Individual bounds will largely be in terms of $\|\Delta v_i\|_1$, the $\ell_1$-norm of the change in values for buyer $i$. Bounds over all agents will mostly use the $\ell_1, \ell_\infty$-norm for matrices, but where the $\ell_1$ is applied to rows rather than columns:

$$\|\Delta V\|_{1, \infty} = \max_{i \in [n]} \|\Delta v_i\|_1.$$  

Note that the standard definition of this norm is to apply it to columns, not rows. Our definition can be thought of as the standard definition, but with a transpose applied before invoking the norm.

While this paper primarily focuses on the divisible setting, our results in this section all carry over to the indivisible setting.

5.1 Individual optimality notions: envy, regret, and maximin share

First we look at individual notions of optimality. These measures will compare how satisfied individuals are under their allocation $\hat{X}$.

We begin with asking how well an equilibrium $(\hat{p}, \hat{X})$ represents buyers’ true preferences. Although in many assignment problems prices are only ‘virtual’ we can still judge the lifted equilibrium by using the abstracted price vector. One notion of how happy individuals are with their allocation $\hat{X}$ is the regret of the allocation. That is, how close is $\hat{x}_i$ to actually being a demand vector given the prices $\hat{p}$? Here we measure regret with respect to the supply constraints when the buyer can buy all items for themselves. Formally the regret of a buyer under a solution $(\hat{p}, \hat{X})$ is

$$\text{Regret}_i(\hat{p}, \hat{X}) = \max\{u_i(x_i) : x_i \in \mathbb{R}^m_+, x_i \cdot \hat{p} \leq B_i, x_i \leq s\} - u_i(\hat{x}_i).$$

By definition, in equilibrium, regret is 0.

The envy for buyer $i$ is the amount by which they prefer any other buyers’ allocation over their own. Equilibrium allocations are envy free in the equal-incomes setting where all buyers have the same
budget. Envy is often used as a fairness metric in the fair-division setting [Varian et al., 1974; Budish, 2011; Caragiannis et al., 2016]. Formally, the envy is

$$\text{Envy}_i(\hat{X}) = \max_{i' \in [n]} u_i(\hat{x}_{i'}) - u_i(\hat{x}_i).$$

Note that the envy gap is zero in this definition, if buyer \( i \) prefers their own bundle.

Finally, we look at a weak notion of fairness: the maximin share (MMS) guarantee. MMS is the value that buyer \( i \) would obtain if they were allowed to choose any allocation \( X \) into \( n \) bundles, but were then assigned their least-valued bundle \( \min_{x' \in [n]} u_i(x_{i'}), \) MMS was introduced by [Budish, 2011], and it generalizes the fair-share notion of divisible assignment to the indivisible setting. In fair-share assignment, each buyer \( i \) is required to receive an allocation that they like at least as much as the one where they receive a fraction \( \frac{1}{n} \) of each item. Since MMS generalizes fair share our result below applies to both fairness notions.

Formally, the MMS guarantee of buyer \( i \) is

$$\text{MMS}_i = \max_{x \in X} \min_{i' \in [n]} u_i(x_{i'}).$$

We say that the MMS gap for buyer \( i \) in allocation \( x \) is

$$\text{MMS gap}_i(\hat{X}) = \max(0, \text{MMS}_i - u_i(\hat{x}_i)).$$

The main result of this section shows that regret, envy, and MMS errors are bounded by the norm of the approximation error.

**Theorem 1** (Regret, Envy, MMS are linear in abstraction quality). The regret, envy, and MMS gap of buyer \( i \) for any allocation \( \hat{X} \) under \( \hat{V} \) is maintained under \( V \) up to individual additive error of \( \|\Delta v_i\|_1 \) for regret and envy, and \( 2\|\Delta v_i\|_1 \) for the MMS gap. The maximum additive error over buyers is thus bounded by \( \|\Delta V\|_{1,\infty} \).

### 5.2 Global optimality notions: Pareto Optimality and Nash Social Welfare

Often in market design the goal is to maximize social welfare, i.e. the sum of buyer utilities. However, in general, a market equilibrium may not lead to a social-welfare-maximizing solution. This is easy to see by considering the solution to the problem of maximizing geometric-mean utility: it is a market equilibrium, but it will never set any buyer utility to zero, even when that is the only way to maximize social welfare.

Instead, the usual efficiency criterion for market equilibria is that they are Pareto optimal (see e.g. Varian et al., 1974; Budish, 2011; Caragiannis et al., 2016). Nonetheless, Pareto optimality can be related to weighted social welfare: Negishi’s theorem (Negishi, 1960) says that given a market equilibrium, there exists a set of utility weights \( \{\beta_i\} \) specifying the utility rate of each buyer such that the market-equilibrium allocation is the allocation that maximizes weighted social welfare. We can use this to show that a solution to \( \hat{V} \) approximately solves the weighted-social-welfare-maximization problem for a particular set of weights:

**Theorem 2** (Weighted social welfare is linear in abstraction quality). Let \( \hat{X} \) be the allocation of a market equilibrium for \( \hat{V} \), and \( \{\beta_i\} \) the associated Negishi social-welfare weights. Then \( \hat{X} \) solves the problem of maximizing social welfare for \( V \) under weights \( \{\beta_i\} \) up to an additive error of at most \( \|\beta\|_1 \|\Delta V\|_{1,\infty} \).

Now we show that any Pareto-improving allocation has some buyer that does not gain much.

**Theorem 3** (Ability to Pareto improve is linear in abstraction quality). Any Pareto-optimal solution under \( \hat{V} \) is such that any Pareto-dominating allocation under \( V \) has a buyer \( i \) whose utility improves by at most \( \|\Delta v_i\|_1 \).

Theorems 2 and 3 show that a solution \( X \) computed for \( \hat{V} \) is approximately Pareto optimal for \( V \) in two senses: It approximately solves the weighted-welfare maximization problem under the Negishi weights from \( \hat{V} \), and there is a small amount of strong Pareto improvement. However, there is a third sense of Pareto improvement which we cannot bound: there may be a Pareto improvement where
some buyer does not gain a lot, but the total improvement to (unweighted) social welfare may be large. We call this largest improvement to unweighted social welfare under any Pareto improvement the \textit{Pareto gap}. Theorem \ref{thm2} guarantees that if the Pareto gap is large, then the large improvements will be for buyers with low Negishi welfare weights, and thus they are, in a sense, buyers whose unweighted utility should not be highly prioritized in market equilibrium.

So far we have discussed global optimality from the perspective of Pareto optimality. However, another global optimality condition that has gained recent interest from the perspective of fairness is Nash social welfare, which is deployed on Spliddit \citep{CGM16}. The Nash social welfare (NSW) is the product of buyer utilities, formally

\[
\text{NSW}(X) = \prod_{i \in [n]} u_i(x_i).
\]

We now show that NSW is also maintained up to multiplicative error in the approximation.

\textbf{Theorem 4} (Nash Social Welfare is multiplicatively related to abstraction quality). The NSW of the optimal solution $\hat{X}$ under $\hat{V}$, is bounded by the NSW of the optimal solution $X^*$ to the original problem as follows:

\[
\text{NSW}(X^*) \leq \prod_{i \in [n]} \left( 1 + \frac{\|\Delta v_i\|_1}{\hat{u}_i(x^*_i)} \right) \text{NSW}(\hat{x}).
\]

This assumes $\hat{u}_i(x^*_i) > 0$ for all $i$, i.e. the difference between $V$ and $\hat{V}$ should be such that the original optimal solution still has nonzero value under $\hat{V}$.

Note that the theorem also implies a bound for the geometric mean of the buyer utilities, which is the $n$'th root of the NSW.

\section{Abstractions in Practice}

The two major obstacles to computing equilibria in practice are information requirements (needing to know every element of $V$) and computation requirements. We now describe two techniques—matrix completion and the use of representative buyers/items—that can be used to reduce these burdens. We discuss how they fit into the framework of abstractions, and relate them to our abstraction bounds derived above. Figure \ref{fig1} summarizes the two abstraction approaches.

\begin{figure}[h]
    \centering
    \includegraphics[width=\textwidth]{figure1.png}
    \caption{Two abstraction methods that can be used together to reduce information and computational constraints. Left: Low rank matrix completion uses observed data to discover a latent vector for each item and buyer. Unobserved valuations are approximated using the dot product of these vectors. Right: Representative buyer/item abstractions collapse multiple buyers and/or multiple items into single representative agents.}
\end{figure}

Recall that most of the bounds (with exception of the Nash social welfare) related the quality of the equilibrium $(\hat{p}, \hat{X})$ to $\|\Delta V\|_{1,\infty}$. We can also replace this norm with the Frobenius norm by using the inequality

\[
\|\Delta V\|_{1,\infty} = \max_{i \in [n]} \|\Delta v_i\|_1 \leq \max_{i \in [n]} \sqrt{m} \|\Delta v_i\|_2 \leq \sqrt{m} \|\Delta V\|_F.
\]

Thus we can replace these bounds by a (less tight) bound of $\sqrt{m} \|\Delta\|_F$. This suggests that when we are looking for ways to abstract our market, we should try to find ways that minimize $\|V - \hat{V}\|_F$. 


6.1 Low Rank Matrix Completion

We begin with the information problem, that is, the case where not all entries of \( V \) are known to us. Without some imputation of the missing entries, we cannot compute an equilibrium. If every \( v_{ij} \) is completely independent and unpredictable from other valuations of the same buyer or other valuations of the same item by other buyers, then we have no hope of filling in missing entries in a sensible way. However, in most real world situations this is likely not the case and there is shared information across entries of the valuation matrix. To fill in the missing entries we can use standard techniques from matrix completion (e.g. (Recht, 2011)).

A standard method for matrix completion is given some observations \( O \) of elements from a matrix \( V \) with generic element \( v_{ij} \) we try to find a set of vectors \( \text{vec} (i) \in \mathbb{R}^d \) for every buyer \( i \) and \( \text{vec} (j) \in \mathbb{R}^d \) for every item \( j \) to solve

\[
\min_{\text{vec}} \sum_{v_{ij} \in O} (v_{ij} - \text{vec}(i) \cdot \text{vec}(j))^2.
\]

After we fit this model, we can construct a now complete matrix \( \hat{V} \) of the original dimensionality with \( \hat{v}_{ij} = \text{vec}(i) \cdot \text{vec}(j) \) and use this in place of \( V \) for our task of interest. The solution ends up effectively minimizing \( \min_{\hat{V}} ||V - \hat{V}||_F \) over rank \( d \) matrices \( \hat{V} \).

6.2 Representative Market Equilibrium

The computation problem of equilibrium comes from the fact that every step of a first order method must solve a maximization problem for each buyer and compare the sum of those demands to the supplies of each item. If an abstraction can reduce the number of buyers (and/or items), it can make the computation of equilibrium much more efficient. Given a valuation matrix \( V \) we consider abstracting the market as follows:

**Algorithm 1** Representative Market Equilibrium

<table>
<thead>
<tr>
<th>Input: ( O ) of known valuations with generic element ( o_{ij} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>procedure CONSTRUCT REPRESENTATIVES</td>
</tr>
<tr>
<td>Get vectors for each buyer ( \text{vec}(i) ) and item ( \text{vec}(j) )</td>
</tr>
<tr>
<td>If ( V ) is fully available, use the row/columns as the vectors</td>
</tr>
<tr>
<td>Use ( k )-means clustering on the buyer vectors with ( \hat{n} ) centroids</td>
</tr>
<tr>
<td>Use ( k )-means clustering on the item vectors with ( \hat{m} ) centroids</td>
</tr>
<tr>
<td>Use centroids of ( k )-means as vector representations of representatives</td>
</tr>
<tr>
<td>Set representative market valuations as dot products of above vectors</td>
</tr>
<tr>
<td>procedure COMPUTE REPRESENTATIVE SUPPLIES/BUDGETS</td>
</tr>
<tr>
<td>Sum budgets of buyers assigned to representative ( i ) by ( k )-means to get budget</td>
</tr>
<tr>
<td>Sum supplies of items assigned to representative ( j ) by ( k )-means to get supply</td>
</tr>
<tr>
<td>Compute market equilibrium in the representative market</td>
</tr>
</tbody>
</table>

6.2.1 Proportional Lift

The algorithm above constructs a representative market equilibrium (RME) which we denote as \( (p_{rep}^*, X_{rep}^*) \). The RME is of a different dimension than the original market. The question is now how to lift the prices and allocations to the original items and buyers. Prices are simple, we can just assign to every item \( j \) the price of its representative item from the RME.

Allocations are more difficult. We first consider a proportional lift. We perform the lift in two steps. First we begin with \( X_{rep}^* \) and construct a new allocation \( X' \) of dimension \( \hat{n} \times \hat{m} \) with \( x'_{ij} \) being the allocation of original items to each representative buyer. Each representative buyer receives a supply weighted share of each item. Let \( r(j) \) be the representative item for real item \( j \). Then the amount of item \( j \) that is allocated to representative buyer \( i \) in this step is

\[
x'_{ij} = \frac{s_j}{\sum_{k \in r(j)} s_k} x_{ir(j)}^*.
\]

Given \( X' \) we finish the lift by now splitting items across individuals proportional to their budget. We denote the final allocation as \( \hat{X} \) with dimension \( n \times m \). Let \( r(i) \) denote the representative buyer for
real buyer $i$. The allocation of item $j$ to real buyer $i$ is
\[ \hat{x}_{ij} = \frac{b_i}{\sum_{k \in r(i)} b_k} x'_{r(i)j}. \]

Because everything is allocated proportionally we have that the supply constraint binds and thus the allocation is a full allocation. The proportional lift has the advantage that it is simple to compute and that our bounds can be applied directly. The allocation from the proportional RME are equivalent to those which would result if we constructed a $\hat{V}$ by getting vector representations for each buyer/item, getting their representative abstraction and replacing $v_{ij}$ by $\text{vec}(r(i)) \cdot \text{vec}(r(j))$.

### 6.2.2 Recursive Lift

The section above divides items assigned to a representative buyer to each of its original buyers proportional to their share of the representative-buyer budget. However, in practice we may wish to perform this lift from representative buyer to original buyer in a smarter way.

We propose doing this by first allocating original items to each representative buyer just as above. We then solve a new market-equilibrium problem independently for each of the $\hat{n}$ representative buyers using the assignments from the proportional allocation as item supplies. The resulting equilibrium allocation for each representative buyer is then used instead of the proportional allocation. Note that we cannot use the prices from these assignments as they are conditional on a subset of the supply, so we use the original representative-item prices. We call this approach recursive representative market equilibrium (RRME).

Note that this method does not give us a straightforward $\hat{V}$ to apply our bounds. However, we can relate various RRME quantities to those that would obtain under the proportional lift (which does have an associated $V$). We now show that RRME is guaranteed to improve on the proportional split on several metrics: Pareto gap, regret, MMS gap, and NSW.

**Theorem 5** (Recursion is Better than Proportional). Let $(\hat{p}, \hat{X})$ be a solution obtained from proportional allocation from a representative market instance. Let $(\hat{p}, X')$ be the solution obtained by applying the recursive-reallocation to the representative market solution. The RRME solution leads to weakly lower Pareto gap, regret, and MMS gap, as well as weakly higher NSW.

We relegate the proof to the Appendix but the intuition for the results comes from the utility guarantee provided by market equilibrium, which says that each buyer does better than their budget-proportional allocation of each item.

Of course, there are tradeoffs to using the recursive lift. The recursive lift requires us to compute $\hat{n}$ more market equilibria. Importantly, though, each of them is considerably smaller than the full problem and since they have no cross dependencies this procedure can be efficiently parallelized. In addition, while the objectives in Theorem 5 are all guaranteed to improve, envy may get worse. On balance, however, the recursive lift is likely better in practice than the proportional one.

**Example 1.** Consider a 5-buyer-4-item instance with valuations $v_1 = v_2 = [1.5, 1.5, 0, 0], v_3 = [0, 0, 1 + \epsilon, 1 - \epsilon], v_4 = [0, 0, 1 - \epsilon, 1 + \epsilon], v_5 = [1.5, 1.5, 1 + \epsilon, 1 - \epsilon]$ and budgets of 1. Now consider the abstraction where buyers 1, 2, 5 are clustered to a representative buyer with valuation $v_1 = [1.5, 1.5, 0, 0]$ and budget 3, and buyers 3, 4 are clustered to a representative buyer with valuation $v_2 = [0, 0, 1.5, 1.5]$ and budget 2. The representative market equilibrium with equal rates to assign representative buyer 1 all of items 1 and 2, each with price 1.5 and assign representative buyer 2 all of 3 and 4 at price 1 each. This assignment leads to no envy when we perform the budget-proportional allocation. However, if we apply RRME then representative buyer 1 still gets proportionally allocated, but for representative buyer 2 buyers 3 and 4 get assigned all of items 3 and 4, respectively. Now buyer 5 envies buyer 3, as they could get $\epsilon$ more utility.

### 7 Experimental Evaluation

#### 7.1 First-Order Methods for Market Equilibrium

We initially attempted to solve the EG convex program directly via the CVXPY package ([Diamond and Boyd, 2016](#Diamond2016)) and ([Akshay Agrawal and Boyd, 2018](#AkshayAgrawal2018)). However, this turned out to have numerical
problems once the number of agents and items reach around 130 each. We tried several solvers including ECOS (Domahidi et al., 2013), CVXOPT (Andersen et al., 2013), and SCS (O’Donoghue et al., 2016).

To solve our scalability issues we propose solving the EG convex program via first-order methods for solving the following Lagrangian relaxation:

\[
\min_{0 \leq p \leq \|B\|_{1/s}} \sum_{i \in [n]} \max_{0 \leq x_i \leq s} B_i \log(v_i \cdot x_i) - p \cdot x_i + s \cdot p
\]

A similar formulation was suggested by Nesterov and Shikhman (2018), where they focus on a more general case (with unbounded prices) and arrive at an algorithm that converges at a rate of \(\frac{1}{\sqrt{T}}\), where \(T\) is the number of iterations.

We can leverage the structure of our problem to provide even better worst-case bounds on convergence. Because in practice such worst case bounds are rarely reached and because these theoretical results are not the primary focus of our paper we discuss them informally here for completeness and relegate the formal statements to the appendix:

**Result 2.** Under mild assumptions we can construct an algorithm that converges to the EG equilibrium at a rate of \(\frac{1}{T}\) iterations.

Because of the structure of our problem we can also give bounds for the computational savings incurred by our representative buyer/representative item abstractions:

**Result 3.** Let \(n, m\) be the number of buyers/items in the original market and \(\hat{n}, \hat{m}\) be the number in the abstracted market. Under mild assumptions the representative buyer/item abstraction with proportional lift has a worst case convergence rate that is \(O(\frac{\hat{n}^2 \hat{m}^2}{n^2 m^2})\) of the convergence rate of the full problem. If we use the recursive lift instead the relevant speedup is \(O(\frac{n^2 m^2}{\hat{n}^2 \hat{m}^2})\).

While we do not take the rates completely seriously, the results do suggest that we should expect larger savings in computational complexity for ‘long and thin’ markets (those with many buyers and relatively few items) than those with many items and few buyers.

### 7.2 General Analysis Plan

We now evaluate the abstraction approaches above on several real datasets. We first discuss the analyses we perform on all datasets then we discuss each dataset in detail and give results.

For each dataset we compute the EG equilibrium in the full market and compare to the prices/allocations we generate using abstractions. We set budgets to 1 and supplies to be such that there is 1 item per person (note that since we are in the divisible case, this only affects the prices/allocations up to a scaling factor).

We vary three properties of abstractions jointly. First, we replace the valuation matrix \(V\) with a rank \(k\) approximation for various \(k\). To compute these low rank representations we use the singular value decomposition (SVD). Second, we consider compressing the large number of buyers into a smaller set of representative agents. We refer to this as abstraction coarseness and measure it in percentage of original market size. Thus, a 40% abstraction is one which replaces the 7200 original buyers with 2800 representative buyers via the \(k\)-means procedure above. Third, we compare the use of a proportional split and the recursive splitting.

We measure all of our theoretical quantities of interest: regret of each individual given each allocation (we normalize this by the maximum utility of the allocation), an individual’s envy (again, normalized by the utility that an individual would receive from the envied bundle), Nash social welfare (normalized by the Nash social welfare of the unabstracted market), Pareto optimality (normalized by the utility in the Pareto optimal allocation), maximin share (normalized by utility being achieved).

We also look at one quantity for which we do not have bounds: total welfare/efficiency of the allocation (normalized by the total welfare of the unabstracted market). Note that market equilibrium makes no pretense of maximizing efficiency (indeed it only guarantees Pareto optimality and, in the case of EG equilibrium, Nash social welfare).
7.3 Dataset: Jester1

We begin by considering an existing dataset used for the evaluation of recommender systems. In typical recommender system datasets, individuals give a rating (rather than a monetary evaluation) for each object. Nevertheless, standard recommender system datasets are still useful examples of valuation matrices for two reasons: first, when equilibrium computation is used for market design/fair division (e.g., CEEI and related algorithms at Wharton or Spliddit), the budget given to each individual is only virtual currency and, second, allocations in EG equilibrium are not affected by the scaling of an individual’s utilities so they are relatively robust to different ways individuals may interpret ratings.

The first dataset we consider, Jester 1, contains the evaluation of 100 jokes by over 79,000 individuals (Goldberg et al., 2001). We extract a submatrix of 7200 individuals that have rated all of the jokes giving us a complete market. Ratings in Jester are on a continuous scale between -10 and 10 and our theory requires positive valuations so we shift the valuations to be strictly positive by shifting the whole matrix by +10.

Figure 2 left panel shows a representation of the Jester1 valuation matrix with lighter colors representing higher valuation. To show the structure more clearly we normalize each individual’s valuations to lie in $[0, 1]$ for the figure (not the experiments) and perform clustering on the rows and columns to set the order they will be displayed. There is very clear block structure suggesting that we can abstract the market using the representative agent method effectively.

The right panel shows our main results. Even a very coarse abstraction with recursion (720 representative buyers for 7200 original buyers and 20% rank compression) can yield an allocation that achieves almost 90% of the Nash social welfare and efficiency of the unabtracted allocation and is almost Pareto optimal (there exists an allocation that improves total utility by 10% without leaving anyone worse off). Individuals display some regret (the abstracted allocation achieves 85% of the utility they could achieve if buyers optimized given the abstraction prices) and some envy. Finally, we see that a weak notion of fair division, maximin share, is almost completely guaranteed under even the coarsest abstraction.

7.4 Dataset: Household Items

We also construct a new dataset of individuals evaluating 50 household items. Unlike in typical recommender datasets where items can often be naturally be clustered into ‘types’ (e.g., romance

\footnote{Recall that since EG equilibrium with budgets is equivalent to equal utility rates per item being received multiplying an individual’s utility by a constant does not affect the equilibrium allocation.}
movies, horror movies) we specifically chose items to span a broad range of product categories (examples: rain jacket, tool box, toaster, shovel, bluetooth headphones, thermos, blackout shade, bike pump, etc... see Appendix for full list of items and average valuations for each one). Items were chosen to be representative of a large number of categories to make our compression problem more difficult.

In an online survey individuals from the US were presented with a photo and brief description of each item. To deal with quality issues specific brands and models for each item were selected from an online review site to be the ‘best in their class’ and participants were informed of this. Items were presented in a random order and participants entered a personal US dollar valuation for each item. At the end of the survey participants were asked how well they felt they understood the questions/task. We use data from the 2876 individuals (out of 3300) that said they felt the task was natural and they could give a good personal valuation for all of the items.

Again, we apply matrix approximation, representative buyer/item modeling, and recursive reallocation. Using the raw data we found that several individuals entered valuations of 0 for 20 or more items and low valuations for others. This means that many allocations gave them a utility of 0 and thus gave a Nash social welfare (product of utilities) of 0. In the results reported here we replace these 0 valuations with 1; we find it does not change any other metric by a meaningful amount but now does allow us to construct meaningful estimates of Nash social welfare.

Figure 3 shows the market, represented as in the Jester1 experiment above (left panel) as well as our main results (right panel). We see that even coarse abstractions can yield good properties and that recursive allocation meaningfully improves Pareto optimality and Nash social welfare of the approximations. There is a sharper tradeoff between compression/low rank approximation and abstraction performance than in Jester. Recall though that this is because our candidate items were specifically chosen to be of quite different types and so display lower inter/intra item correlations in valuation than typical recommender datasets of e.g. jokes or movies.

### 7.5 Dataset: MovieLens

Both datasets above includes many more buyers than items. We would like to consider a dataset where both sides of the market are large. However, asking individuals to evaluate many hundreds of items would be expensive and likely yield low quality data. Instead, we construct such a market from MovieLens 1M [Harper and Konstan, 2016], a standard dataset for the evaluation of matrix completion algorithms. This dataset contains 6040 individuals with their ratings for a selection of 3952 total movies. In total, the dataset includes 1 million ratings. We treat each 1-5 rating as a valuation.
MovieLens 1M is very sparse (only a small percentage of possible user/movie rating pairs are observed) so we cannot compute the ground truth market equilibrium as we could in the other datasets. To get around this we use this data to construct a complete market. We compute a low rank approximation to the true rating matrix using standard methods (see Appendix for full details) and consider the submatrix which consists of the 1500 movies with the most observed ratings (avg. number of ratings: 579) and the top 1500 users who have rated the most movies (avg. number of ratings: 425). We use this submatrix as our valuation matrix for the market (Figure 4 left panel shows the market).

Because the completion method already uses a low rank approximation and because we cannot get the ground truth market equilibrium we only test the representative agent abstraction here. Unlike in the datasets above we construct both representative buyers and representative items. Figure 4 right panel, shows that in this case even an extremely coarse abstraction (150 representative buyers, 150 representative items instead of the original 1500) can achieve outcomes that are quite close those of the full equilibrium.

Figure 4: Market created by using valuation matrix from the MovieLens dataset with 1500 buyers and 1500 items (left panel). There is clear structure suggesting compressibility. In this experiment we apply both representative buyer and representative agent abstraction. Equilibria computed in quite coarse abstractions maintain good properties (right panel).

8 Conclusion and Future Directions

Computing market equilibria is a difficult problem. We have shown that the method of abstraction - solving a coarser problem and lifting the solution - can be used to reduce the information requirements and computation requirements of equilibrium computation. In addition, we have introduced a new dataset that we hope others can use for work on fair division.

There are many future directions to expand this research. In this work we looked at Fisher markets which assume additive valuations. This rules out situations where goods can be complements or substitutes. In the case of such preferences the information required to compute a market equilibrium becomes extremely large as we need to know the valuations of individuals for every possible combination of goods (Porter et al., 2003). However, recent work has begun to explore representing preferences with complements and substitutes in low rank vector format (Ruiz et al., 2017; Peysakhovich and Ugander, 2017). An interesting future direction is combining such techniques with abstraction methods. Generalizing beyond linear valuations could lead to greater scalability for problems such as public decision making, which have recently been related to market equilibria (Garg et al., 2018).

We considered the use of market equilibria as allocation mechanisms (as in e.g. the literature on fair division). However, another important use of computation of market equilibria is counterfactual estimation as in structural models in economics (Berry et al., 1995; Chawla et al., 2017). For example, an online marketplace may want to know how prices (and thus revenues) would change if certain
market conditions (e.g. supplies, budgets) were to change. Using the method of abstracting large markets to answer such questions is also an important future direction.

We looked at two methods that could be used in concert for abstraction creation: low rank approximation and representative buyer/item modeling. The representative market abstraction speeds up computation but we did not use low rank structure for anything but filling in missing data. An interesting algorithmic question is whether the low rank structure can be leveraged to speed up the gradient calculation steps of our first order methods, for example, by employing recent techniques for fast nearest neighbor search [Johnson et al., 2017].

Our work fits into the nascent but growing literature on combining techniques from machine learning/AI with classical results from game theory to solve market design [Feng et al., 2018; Golowich et al., 2018], game abstraction [Moravčík et al., 2017; Brown et al., 2018], and agent design [Lerer and Peysakhovich, 2017, 2019] problems that cannot be easily solved in closed form. In this work we leveraged very standard linear abstractions (low rank approximation, k-means clustering). A question for future work is whether more complex, non-linear methods, can be used to construct even better abstractions.

References


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9 Appendix

9.1 Proofs of Theorems

**Proof of Lemma** [7] The first inequality follows from linearity of valuations along with the definition of $\Delta V'_i$:

$$v_i \cdot x_i + \epsilon = \hat{v}_i \cdot x_i + \Delta v_i \cdot x_i + \epsilon \geq \hat{v}_i \cdot x'_i + \Delta v_i \cdot x_i = v_i \cdot x'_i + \Delta v_i \cdot (x_i - x'_i).$$

The second inequality follows by observing that $x'_i - x_i \in [-1, 1]^m$ and thus $\Delta v_i \cdot (x'_i - x_i) \leq \|\Delta v_i\|_1$.

**Proof of Theorem** [7] Individual Optimality Results. First we show the regret bound. Let $\hat{r}_i$ be the regret bound for each buyer $i$ under $\hat{V}$. By the definition of bounded regret we have that for all $i$ Lemma [1] is satisfied for any budget-and-supply-feasible $\hat{x}'_i$ with $\epsilon = \hat{r}_i$, thus bounding regret under $V$ by $\hat{r}_i + \|\Delta v_i\|_1$.

Next we show the envy bound. By the definition of the envy gap we have that for all $j \neq i$ Lemma [1] is satisfied for the pair $\hat{x}_i, \hat{x}_j$ with $\epsilon = \text{Envy gap}_i$. The result follows immediately by taking the maximum over buyers.

Finally we show the MMS bound. Let $X$ be an allocation obtaining the MMS guarantee. It follows that for all $i' \in [n]$

$$\text{MMS}_i \leq v_i \cdot x_{i'} = (\hat{v}_i + \Delta v_i) \cdot x_{i'} \leq \hat{v}_i \cdot x_{i'} + \|\Delta v_i\|_1.$$

Thus when making bundles under $\hat{V}$ buyer $i$ can choose $X$ and achieve at least $\text{MMS}_i - \|\Delta v_i\|_1$, which shows that the MMS guarantee for buyer $i$ under $\hat{V}$ is at least $\text{MMS}_i - \|\Delta v_i\|_1$. Now let $\hat{X}$ be an allocation with MMS gap $\epsilon_i$ under $\hat{V}$; this can similarly be bounded in order to show the result:

$$\text{MMS}_i - \|\Delta v_i\|_1 \leq \hat{v}_i \cdot \hat{x}_i + \epsilon_i = (v_i - \Delta v_i) \cdot \hat{x}_i + \epsilon_i \leq v_i \cdot \hat{x}_i + \|\Delta v_i\|_1 + \epsilon_i.$$

**Proof of Theorem** [7] bounded improvement to Negishi-weighted social welfare. By Negishi’s welfare theorem $\hat{X}$ is such that $\sum_{i \in [n]} \beta_i \hat{v}_i \cdot \hat{x}_i \geq \sum_{i \in [n]} \beta_i \hat{v}_i \cdot \hat{x}'_i$ for all feasible allocations $\hat{X}'$. 

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Thus we get
\[
\sum_{i \in [n]} \beta_i v_i \cdot \hat{x}_i = \sum_{i \in [n]} \beta_i (\hat{v}_i + \Delta v_i) \cdot \hat{x}_i \\
\geq \sum_{i \in [n]} \beta_i \hat{v}_i \cdot \hat{x}_i + \sum_{i \in [n]} \beta_i \Delta v_i \cdot \hat{x}_i \\
= \sum_{i \in [n]} \beta_i v_i \cdot \hat{x}_i' + \sum_{i \in [n]} \beta_i \Delta v_i \cdot (\hat{x}_i - \hat{x}_i') \\
\geq \sum_{i \in [n]} \beta_i v_i \cdot \hat{x}_i' - \sum_{i \in [n]} \beta_i \|\Delta v_i\|_1 \\
\geq \sum_{i \in [n]} \beta_i v_i \cdot \hat{x}_i' - \|\beta_i\|_1 \|\Delta V\|_{1,\infty}
\]

\[\square\]

**Proof of Theorem 3** small strong Pareto improvement. Let \(\hat{X}\) be any Pareto-optimal solution under \(\hat{V}\) and assume that \(\hat{X}\) is not Pareto optimal under \(\hat{V}\). Let \(X\) be the Pareto-improving allocation. Since \(\hat{X}\) was Pareto optimal under \(\hat{V}\) we know that there exists some buyer \(i\) such that the condition of Lemma 1 holds with \(\epsilon = 0\) for that buyer. This immediately implies the theorem.

**Proof of Theorem 4** Bounded NSW. We have
\[\text{NSW}(X^*) = \prod_{i \in [n]} v_i \cdot x_i^* = \prod_{i \in [n]} (\hat{v}_i \cdot x_i^* + \Delta v_i \cdot x_i^*) = \prod_{i \in [n]} \hat{v}_i \cdot x_i^* \left(1 + \frac{\Delta v_i \cdot x_i^*}{\hat{v}_i \cdot x_i^*}\right)\]
Now we can use the fact that \(X^*\) is feasible under \(\hat{V}\) to note that its value must be less than that of \(\hat{X}\):
\[\leq \prod_{i \in [n]} \hat{v}_i \cdot \hat{x}_i \left(1 + \frac{\Delta v_i \cdot x_i^*}{\hat{v}_i \cdot x_i^*}\right) \leq \prod_{i \in [n]} \hat{v}_i \cdot \hat{x}_i \left(1 + \frac{\|\Delta V\|_1}{\hat{v}_i \cdot x_i^*}\right) = \text{NSW}(\hat{X}) \prod_{i \in [n]} \left(1 + \frac{\|\Delta V\|_1}{\hat{v}_i \cdot x_i^*}\right).\]

\[\square\]

**Proof of Theorem 5** RME weakly improves buyer utilities. First we note the following simple fact which holds for any buyer \(i\): the utility of \(i\) under \(X^*\) is weakly greater than that under \(X\), i.e. \(v_i \cdot x_i^* \geq v_i \cdot x_i\). This is because (a subset of) \(X^*\) is a market equilibrium in the recursive market for the corresponding \(i\), and a buyer is guaranteed to get at least the value of the budget-proportional allocation in any market equilibrium. The Pareto gap is the value of a linear program that maximizes social welfare (minus current welfare) subject to the constraint that each buyer is weakly better off. Since utilities are greater in \(X^*\) this is a strictly more constrained problem than for \(X\), and thus the value, i.e. the Pareto gap, is lower. Since we keep prices the same the optimal bundle \(x_i^*\) for each buyer remains the same for \((X, p)\) and \((X^*, p)\). Thus the only affected part of regret is the negative term, which is weakly greater under \(X^*\) since utilities are weakly greater. That the MMS gap is smaller and NSW greater follows directly from each buyer having weakly-higher utility.

\[\square\]

### 9.2 Convergence Rates of RME Computation

Formulation \(\smp(1)\) allows us to apply standard algorithms for solving convex-concave saddle-point problems such as the primal-dual algorithm of Chambolle and Pock \(2016\) \(2011\) (henceforth referred to as PD). PD is an algorithm for solving problems of the following form (we omit some generality which we do not need):
\[\min_{x \in X} \max_{p \in P} \mathcal{L}(x, p) := x^T Kp + f(x) + s \cdot p \tag{2}\]
where $K$ is a matrix with bounded norm $L = \|K\| = \max_{\|x\| \leq 1, \|p\| \leq 1} x^T K p$, and $f$ is a proper, lower semicontinuous convex function with a Lipschitz-bounded gradient, i.e. $\|\nabla f(x) - \nabla f(x')\|_2 \leq L_f \|x - x'\|_2$ for all $x, x' \in \mathcal{X}$.

An iteration of PD is as follows:

$$D(t) = \text{arg min}_{p \in P} \frac{1}{\tau} \|x - \bar{x}\| - \bar{x}^T K \bar{p}$$

The logarithm in the objective function presents a challenge because the gradient is unbounded near zero; this problem can be addressed by noting that agents are always guaranteed to receive their MMS value in equilibrium, and thus we can add the additional constraint $v_i \cdot x_i \geq \text{MMS}$ to the feasible set of each agent, thereby bounding the gradient difference $L_f$ by $\max_{i \in [n], j \in [m]} v_i B_j$. In practice we found that utilities did not approach zero and thus this projection was unnecessary.

We now show how to instantiate our SPP (1) in terms of the generic SPP (2). We have that

$$\mathcal{D}(x, \bar{x}) = \mathcal{D}(x, \bar{x}) + \langle \nabla f(\bar{x}), x - \bar{x} \rangle + x^T K \bar{p} + \frac{1}{\tau} \|x - \bar{x}\|$$

The average iterates $\bar{x} = \sum_{t=1}^{T} x_t$ and $\bar{p} = \sum_{t=1}^{T} p_t$ converge to a saddle-point solution at a rate of $O((L + L_f) T^{-\frac{1}{2}} + L D^2)$, where $D_x, D_p$ is the maximum value of the $\ell_2$ norm over $\mathcal{X}$ and $P$ respectively. Note that we upper bound the price vector $p$ by the sum of budgets $\|B\|_1$, and each allocation vector $x_i$ by the supply of each item. This does not change the set of equilibria, as these conditions are all guaranteed to be satisfied in equilibrium.

We now show how to instantiate our SPP (1) in terms of the generic SPP (2). We have that $\mathcal{X}$ is the product of allocation vectors $\times_{i \in [n]} \{x_i : 0 \leq x_i \leq s\}$ over the buyers, and $P = \{p : 0 \leq p \leq \|B\|_1/s\}$ is the set of price vectors. $K$ is an $nm \times m$ matrix representing $\sum_{i \in [n]} p \cdot x_i$, i.e. with a 1 in each row/column-pair $r, j$ when the row $r$ corresponds to a variable that denotes assigning item $j$ to some bidder. The norm $L$ of $K$ is $\sqrt{n}$, which is achieved by any pair $(x, p)$ such that $p_j = 1$ for some $j$, and $x_{ij} = \frac{1}{\sqrt{n}}$, with all other entries 0, or by setting $p_j = \frac{1}{\sqrt{m}}, x = \frac{1}{\sqrt{mn}}$.

The logarithm in the objective function presents a challenge because the gradient is unbounded near zero; this problem can be addressed by noting that agents are always guaranteed to receive their MMS value in equilibrium, and thus we can add the additional constraint $v_i \cdot x_i \geq \text{MMS}$ to the feasible set of each agent, thereby bounding the gradient difference $L_f$ by $\max_{i \in [n], j \in [m]} v_i B_j$. In practice we found that utilities did not approach zero and thus this projection was unnecessary.

The value of $D_x$ is $n \|s\|^2_2$, the maximum is achieved by setting $x_{ij} = s_j$ for all $i, j$. The value of $D_p$ is $\sum_{j \in [m]} (\|B\|_1/s_j)^2$, the maximum is achieved by setting each price at its upper bound.

Putting together this construction gives an algorithm that converges to a saddle point of (1) at a rate of

$$\max_{p \in P} \mathcal{L}(\bar{x}, p) - \min_{x \in \mathcal{X}} \mathcal{L}(x, \bar{p}) \leq O\left(\frac{\sqrt{n} + \max_{i \in [n]} v_i B_i}{\text{MMS}} \|s\|^2_2 + \sqrt{n} \sum_{j \in [m]} (\|B\|_1/s_j)^2\right).$$

Now we see that if we solve a clustering $\{\{C_i\} : \{C_j\}\}$ we get the following convergence rate:

$$O\left(\frac{\sqrt{n} + \max_{i \in [n], j \in [m]} \frac{\delta_i \bar{n}}{\text{MMS}} \|s\|^2_2 + \sqrt{n} \sum_{j \in [m]} (\|B\|_1/s_j)^2\right).$$ (3)

Furthermore, each iteration is of order $O(n \bar{n})$ rather than $O(n m)$. To get a sense of the savings in running time, say that supply and budgets are both 1. In that case the original problem has runtime cost $O(n^{7/2} m^2)$ (since the third term usually dominates), and thus compressing to 10% problem size leads to a runtime decrease of factor 100 – 3162 depending on whether the decrease in instance size is primarily due to fewer items or buyers.

If we apply recursive lift then we have to solve $k$ market equilibrium problems corresponding to each cluster of buyers. The cost of computing the recursive allocations is the sum of the costs of computing each recursive market equilibrium. This can be expressed as a sum over terms similar to (3), but where $\bar{n}, \bar{n}$ represent the size of the given recursive instance. Again we assume that budgets and supplies are 1 to get a sense of runtime savings. We can then use the fact that the union of the buyer clusters is $[n]$ to bound the runtime as $O(n \bar{n}^{7/2} \bar{n}^2)$, where $\bar{n}, \bar{n}$ corresponds to the size of
the recursive market equilibrium which maximizes $\hat{\eta}^{5/2} \hat{\mu}^2$. Thus even with the additional cost of computing the RME allocation the runtime cost savings are on the order of $100 - 316$ for the case where each recursive market is $\frac{1}{10}$ the size of the original market.

9.3 Details of MovieLens Market Generation

To generate the complete submatrix for the MovieLens 1M market we take the observed ratings denoted as $O$ with generic element $o_{ij}$. We use PyTorch (Paszke et al., 2017) and minimize the loss function

$$\sum_{o_{ij} \in O} (o_{ij} - \text{vec}(user_i) \cdot \text{vec}(movie_j) + \text{bias}(user_i) + \text{bias}(movie_j))^2$$

over $d$ dimensional vectors for each user and movie and 1 dimensional biases for each user and movie. We random split the data into an 80% training set and a 20% validation set and cross-validate the choice of $d$ from the set \{20, 30, 50, 70, 100\} and the weight decay parameter from the set \{1e-5, 1e-4, 1e-3, 1e-2, 1e-1\}. We choose the best performing model via the validation set ($d=20$, weight decay=1e-5) which yields an RMSE of $.88$ which is comparable with other matrix factorization approaches. Though we point out that more complex approaches, e.g. autoencoder-based ones do outperform these models (Sedhain et al., 2015).

9.4 Details of Household Dataset

To construct the dataset we recruited US-based workers from an online labor market. We chose 50 items from a well regarded online review site and presented each one with each item asking them to give their ‘person valuation’ for the item. Here we list the items we used as well as their average valuations by individuals.

Figure 5: Items used in household item survey as well as their average valuations and the standard deviations of the reported valuations.